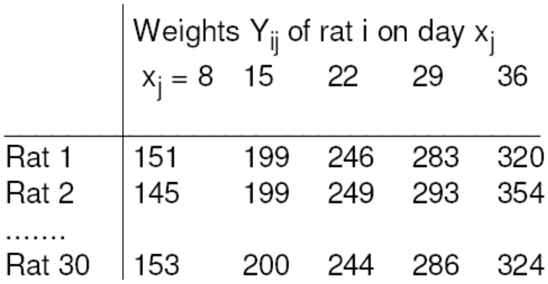
# Lecture 4
# Linear random coefficients models

## Rats example

- 30 young rats, weights measured weekly for five weeks
- Dependent variable ($Y_{ij}$) is weight for rat “i” at week “j”
- Data:



| | Weights $Y_{ij}$ of rat i on day $x_j$ | | | | |
|---|---|---|---|---|---|
| | $x_j = 8$ | 15 | 22 | 29 | 36 |
| Rat 1 | 151 | 199 | 246 | 283 | 320 |
| Rat 2 | 145 | 199 | 249 | 293 | 354 |
| ……. | | | | | |
| Rat 30 | 153 | 200 | 244 | 286 | 324 |

- Multilevel: weights (observations) within rats (clusters)

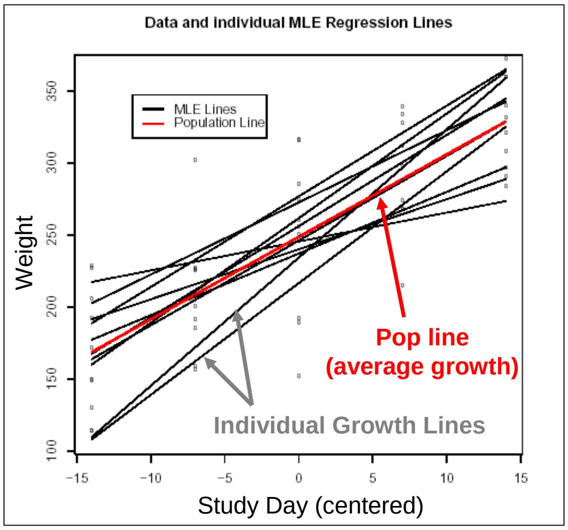
# Individual & population growth

- Rat "i" has its own expected growth line:

$$E[Y_{ij} \mid b_{0i}, b_{1i}] = b_{0i} + b_{1i} x_j$$

- There is also an overall, average population growth line:

$$E[Y_{ij}] = \beta_0 + \beta_1 x_j$$



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# Improving individual-level estimates

- Possible Analyses

1. Each rat (cluster) has its own line:

   intercept= $b_{i0}$, slope= $b_{i1}$

2. All rats follow the same line:

   $b_{i0} = \beta_0 \quad , \quad b_{i1} = \beta_1$

3. A compromise between these two:

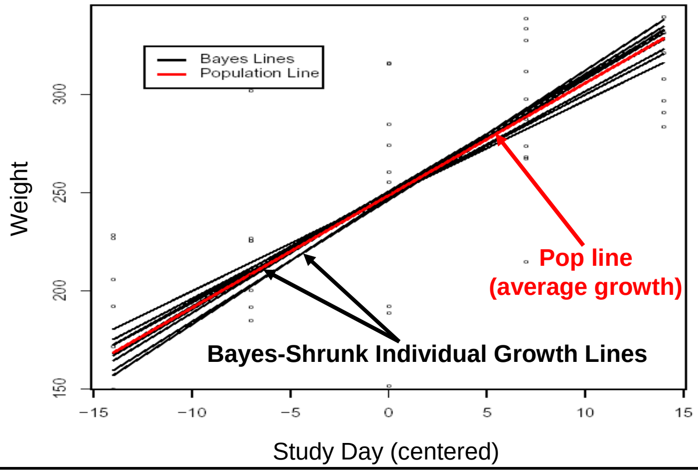
   Each rat has its own line, BUT…

   the lines come from an assumed distribution

$$E(Y_{ij} \mid b_{i0}, b_{i1}) = b_{i0} + b_{i1}X_j$$

$$\text{“Random Effects”} \begin{cases} b_{i0} \sim N(\beta_0 , \tau_0^2) \\ b_{i1} \sim N(\beta_1 , \tau_1^2) \end{cases}$$

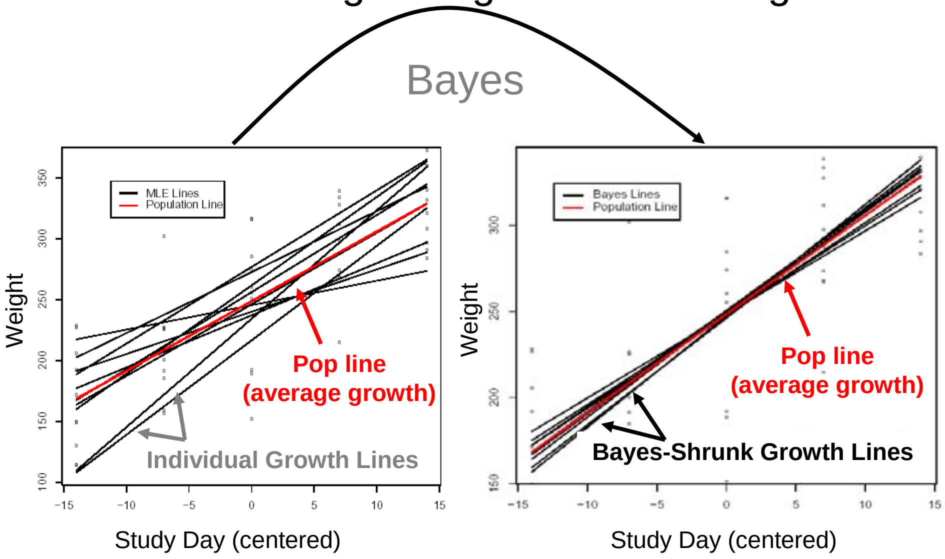
## A compromise:
## Each rat has its own line, but information is borrowed across rats to tell us about individual



Pop line
(average growth)

Bayes-Shrunk Individual Growth Lines

## Bayesian paradigm provides methods for "borrowing strength" or "shrinking"



Pop line
(average growth)

Individual Growth Lines

Pop line
(average growth)

Bayes-Shrunk Growth Lines

## Inner-London School data:
## How effective are the different schools?
## (gcse.dat,Chap 3)

- Outcome: score exam at age 16  (gcse)
- Data are clustered within schools
- Covariate: reading test score at age 11 prior enrolling in the school (lrt)
- Goal: to examine the relationship between the score exam at age 16 and the score at age 11 and to investigate how this association varies across schools

## Fig 3.1: Scatterplot of gcse vs lrt for school 1 with regression line)

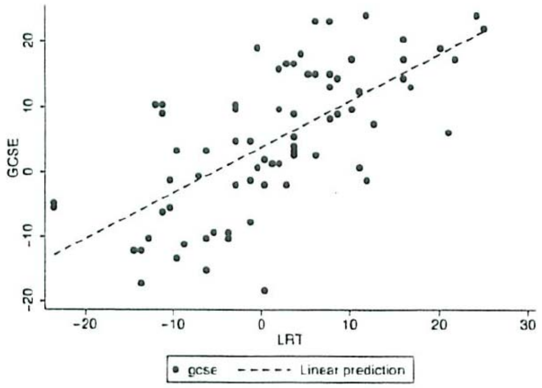


Figure 3.1: Scatterplot of `gcse` versus `lrt` for school 1 with regression line

## Linear regression model with random intercept and random slope

centered

$$Y_{ij} = b_{0j} + b_{1j} x_{ij} + \varepsilon_{ij}$$

$$b_{0j} \sim N(\beta_0, \tau_1^2)$$

$$b_{1j} \sim N(\beta_1, \tau_2^2)$$

$$\mathrm{cov}(b_{0j}, b_{1j}) = \tau_{12}$$

## Alternative Representation
## Linear regression model with random intercept and random slope

$$Y_{ij} = b_{0j} + \beta_0 + (b_{1j} + \beta_1) x_{ij} + \varepsilon_{ij}$$

$$b_{0j} \sim N(0, \tau_1^2)$$

$$b_{1j} \sim N(0, \tau_2^2)$$

$$\mathrm{cov}(b_{0j}, b_{1j}) = \tau_{12}$$

## Fig 3.3: Scatterplot of intercepts and slopes for all schools with at least 5 students

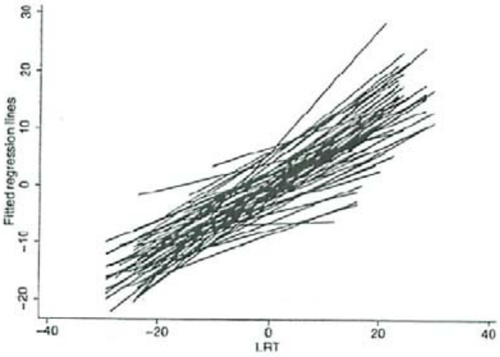


Figure 3.3: Scatterplot of intercepts and slopes for all schools with at least 5 students

## Linear regression model with random intercept and random slope

$$Y_{ij} = (b_{0j} + \beta_0) + (b_{1j} + \beta_1)x_{ij} + \varepsilon_{ij}$$

$$Y_{ij} = (\beta_0 + \beta_1 x_{ij}) + (b_{0j} + b_{1j}x_{ij}) + \varepsilon_{ij}$$

$$\xi_{ij} = (b_{0j} + b_{1j}x_{ij}) + \varepsilon_{ij}$$

$$\text{var}(\xi_{ij}) = \tau_1^2 + 2\tau_{12}x_{ij} + \tau_2^2 x_{ij}^2 + \sigma^2$$

The total residual variance is said to be heteroskedastic because depends on x

$\tau_2^2 = \tau_{12} = 0$ Model with random intercept only

$$\text{var}(\xi_{ij}) = \tau_1^2 + \sigma^2$$

# Empirical Bayes Prediction (xtmixed reff*,reffects)

In stata we can calculate:

$(\tilde{b}_{0j}, \tilde{b}_{1j})$ EB: borrow strength across schools

$(\hat{b}_{0j}, \hat{b}_{1j})$ MLE: DO NOT borrow strength across Schools

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| | Random Intercept | | Random Intercept and Slope | |
|---|---|---|---|---|
| | Est | SE | Est | SE |
| _cons | 0.02 | 0.40 | -0.12 | 0.40 |
| lrt | 0.56 | 0.01 | 0.56 | 0.02 |
| Random | | | | |
| xtmixed | | | | |
| Tau_11 | 3.04 | 0.031 | 3.01 | 0.30 |
| Tau_22 | | | 0.12 | 0.02 |
| Rho_21 | | | 0.50 | 0.15 |
| Sigma | 7.52 | 0.84 | 7.44 | 0.08 |
| gllamm | | | | |
| Tau_11^2 | 9.21 | 1.83 | 9.04 | 1.83 |
| Tau_22^2 | | | 0.01 | 0.00 |
| Tau_21 | | | 0.18 | 0.07 |
| Sigma^2 | 56.57 | 1.27 | 55.37 | 1.25 |

Correlation between random effects → Rho_21

Between Schools variance → Tau_11^2, Tau_22^2

Within school variance → Sigma^2

# Fig 3.9: Scatter plot of EB versus ML estimates

The resulting graphs are shown in figure 3.9.

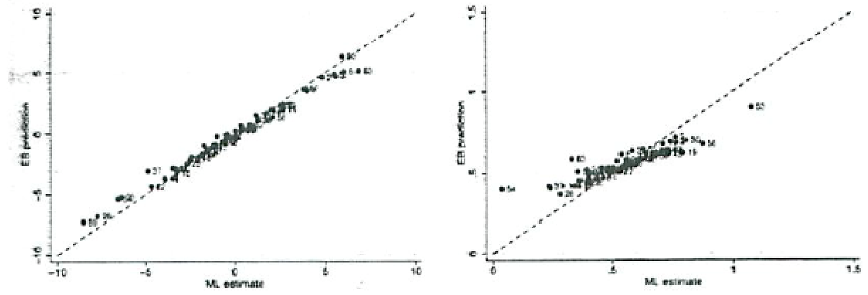

Figure 3.9: Scatterplot of EB predictions versus ML estimates of school-specific intercepts (left) and slopes (right) with equality shown as reference lines

# Fig 3.10: EB predictions of school-specific lines

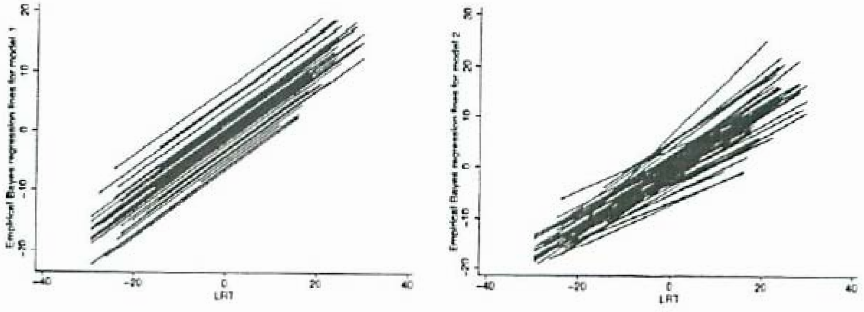

Figure 3.10: Empirical Bayes predictions of school-specific regression lines for the random-intercept model (left) and the random-intercept and random-slope model (right)

# Random Intercept EB estimates and ranking (Fig 3.11)

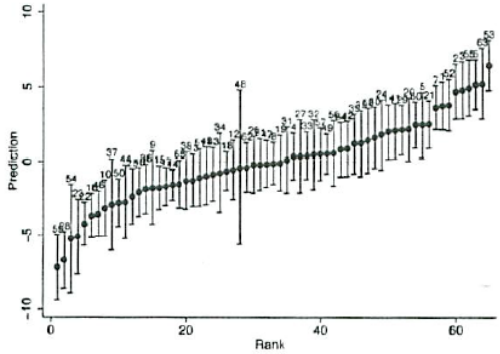

Figure 3.11: Random-intercept predictions and approximate 95% confidence intervals versus ranking (school identifiers shown on top of confidence intervals)

# Growth-curve modelling (asian.dta)

•Measurements of weight were recorded for children up to 4 occasions at 6 weeks, and then at 8,12, and 27 months

•Goal: We want to investigate the growth trajectories of children's weights as they get older

•Both shape of the trajectories and the degree of variability are of interest

## Fig 3.12: Observed growth trajectories for boys and girls

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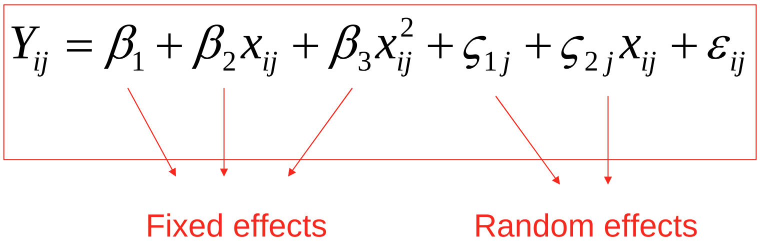
# What we see in Fig 3.12?

- Growth trajectories are not linear
- We will model this by including a quadratic term for age
- Some children are consistent heavier than others, so a random intercept appears to be warranted

## Quadratic growth model with random intercept and random slope



$$Y_{ij} = \beta_1 + \beta_2 x_{ij} + \beta_3 x_{ij}^2 + \varsigma_{1j} + \varsigma_{2j} x_{ij} + \varepsilon_{ij}$$

Fixed effects

Random effects

Random effects are multivariate normal with means 0, standard deviations tau_11 and tau_22 and covariance tau_12

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## Results for Quadratic Growth Random Effects Model

| | Random Intercept | | Random Intercept and Slope | |
|---|---|---|---|---|
| | Est | SE | Est | SE |
| _cons | 3.43 | 0.18 | 3.49 | 0.14 |
| Age | 7.82 | 0.29 | 7.70 | 0.24 |
| Age^2 | -1.71 | 0.11 | -1.66 | 0.09 |
| | | | | |
| Random | | | | |
| Tau_11 | 0.92 | 0.10 | 0.64 | 0.13 |
| Tau_22 | | | 0.50 | 09.09 |
| Rho_21 | | | 0.27 | 0.33 |
| Sigma | 0.73 | 0.05 | 0.58 | 0.05 |

Random intercept standard deviation

**Level-1 residual standard deviation**

Correlation between baseline and linear random effects….

# Two-stage model formulation

$$y_{ij} = \eta_{1j} + \eta_{2j}x_{ij} + \beta_3 x_{ij}^2 + \varepsilon_{ij}$$

Stage 1

$$\eta_{1j} = \gamma_{11} + \gamma_{12}Girl_j + \varsigma_{1j}$$

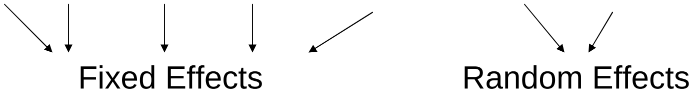
$$\eta_{2j} = \gamma_{21} + \gamma_{22}Girl_j + \varsigma_{2j}$$

Stage 2

$$y_{ij} = \gamma_{11} + \gamma_{12}Girl_j + \varsigma_{1j} + \gamma_{21}x_{ij} + \gamma_{22}Girl_j x_{ij} + \varsigma_{2j}x_{ij} + \beta_3 x_{ij}^2 + \varepsilon_{ij}$$

$$y_{ij} = \gamma_{11} + \gamma_{21}x_{ij} + \beta_3 x_{ij}^2 + \gamma_{12}Girl_j + \gamma_{22}Girl_j x_{ij} + \varsigma_{1j} + \varsigma_{2j}x_{ij} + \varepsilon_{ij}$$

$$y_{ij} = \beta_0 + \beta_1 x_{ij} + \beta_2 x_{ij}^2 + \beta_3 Girl_j + \beta_4 Girl_j x_{ij} + \varsigma_{1j} + \varsigma_{2j}x_{ij} + \varepsilon_{ij}$$



Fixed Effects

Random Effects

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# Results from Random intercept and slope model with and without inclusion of gender effect

| | Random Intercept and Slope | | Random Intercept and Slope | |
|---|---|---|---|---|
| | Est | SE | Est | SE |
| _cons | 3.49 | 0.14 | 3.75 | 0.17 |
| Age | 7.70 | 0.24 | 7.81 | 0.25 |
| Age^2 | -1.66 | 0.09 | -1.66 | 0.09 |
| Girl | | | -0.54 | 0.21 |
| Girl*Age | | | -0.23 | 0.17 |
| | | | | |
| Random | | | | |
| Tau_11 | 0.64 | 0.13 | 0.59 | 0.13 |
| Tau_22 | 0.50 | 09.09 | 0.50 | 0.09 |
| Rho_21 | 0.27 | 0.33 | 0.19 | 0.34 |
| Sigma | 0.58 | 0.05 | 0.57 | 0.05 |

## More on interpreting results

- See handout!