Bayesian inference

The main idea of Bayesian analysis is easy to state. Suppose p is the proportion of voters in Florida who intend to vote for Bush. Note that this isn't quite the same p that appeared earlier; that p was a *probability*, not a proportion. In this section, if I say $p = .5$ I mean that exactly half of the votes go to Bush, and the election is a tie.

Using the binomial theorem, we find that the probability of obtaining 366 votes for Bush and 303 for Kerry in our sample of 669 voters is

$$\binom{669}{303} p^{366}(1-p)^{303}.$$

We can thus compute that, if $p = .55$, the chance that a random sample of 669 voters would contain 336 Bush supporters is about 3 percent. If p = .5, we should expect it to be less likely that the sample will come out so strongly in favor of Bush; and indeed, we find that when $p = .5$ the chance of obtaining a $366 - 303$ margin in our sample is just 0.0016, about 20 times less likely. Bayes's theorem allows us to turn this analysis on its head, saying: given that the poll came out $366 - 303$, it is 20 times more likely that $p = .55$ than that $p = .5$. Applying this argument to all possible values of p eventually allows us to specify precisely the probability that p assumes any particular value. Now if you don't want to see equations, don't read any further!

But if you do: suppose A and B are two events, and let $P(A)$ be the probability that A occurs, $P(B)$ the probability that B occurs, $P(A, B)$ the probability that both A and B occur, and $P(A|B)$ the probability that A occurs given that B occurs. Then Bayes's theorem says

$$P(A|B) = P(A, B)/P(B).$$

In this case, we can take B to be the event "The poll results are 366-303 in favor of Bush" and A to be the event "The voters in Florida are evenly split, 3 million for Bush and 3 million for Kerry." What we are trying to compute is $P(A|B)$; the probability that the vote is a tie, given the 366-303 poll result. What we *know* about is $P(B|A)$, the probability that we'll obtain a 366-303 poll result given equal numbers of Bush and Kerry voters in Florida; this probability, as mentioned above, is about 0.0016.

For each number n between 0 and 6000000, write A_n for the event "The proportion of voters in Florida supporting Bush is n." Then we've already seen that

$$P(B|A_n) = \binom{669}{303} p^{366}(1-p)^{303}.$$

where $p = n/6000000$ is the proportion of voters supporting Bush. And, since we've stipulated that our prior belief holds all values of n to be equally likely, we can say $P(A_n) = 1/6000000$ for all n; so by Bayes's theorem

$$P(B, A_n) = P(B|A_n)P(A_n) = (1/6000000)P(B|A_n).$$

Now I claim

$$P(B) = \sum_{n=0}^{6000000} P(B, A_n);$$

this amounts to saying "the chance that the poll comes out 366-303 is obtained by summing, over all possible n, the chance that the poll comes out $366 - 303$ and that n voters in Florida support Bush." This sum is just

$$(1/6000000)\binom{699}{303}\sum_{n=0}^{6000000}(n/6000000)^{366}(1-n/6000000)^{303}$$

which comes out to about 0.0149. Now we can return to our original problem: using Bayes's theorem again, we have

$$P(B|A) = P(B, A)/P(A) = 6000000P(B, A).$$

Whence

$$P(A|B) = P(B, A)/P(B) = P(B|A)/(6000000P(B)).$$

Since $P(B|A) = 0.0016$ and $P(B) = 0.0149$, we conclude that

$$P(A|B) = 1.77 \times 10^{-7}$$

which is to say that, given the 366-303 poll result, the chance of a perfect tie is about 1 in 5000000, as claimed.