Derivation of the full conditionals for Bayesian analysis

In the following, we derive the straightforward calculations of the full conditionals for Bayesian analysis. For the Bayesian model specified in the paper, the likelihood is given by:

$$
\begin{aligned}
& f(y, d_w, m, u, d_u, \mu_u, \sigma^2_{u,d_u}, \pi_u, \beta^c) \\
= & \prod_{i=1}^{I} \prod_{j=1}^{J} f(y_{ij}, d_{w,ij}, m_{ij}, u_i, d_{u,i}, \mu_u, \sigma^2_u, \pi_u, \beta^c) \\
= & \prod_{i=1}^{I} \left[\prod_{j=1}^{J} \left\{ \right. \right. \\
& \prod_{l=1}^{L_w} \pi_{w,l}^{I(d_{w,ij}=l)} \\
& \times \frac{1}{(2\pi\sigma^2_{w,d_{w,ij}})^{1/2}} \exp\left\{ -(m_{ij} - \mu_{w,d_{w,ij}} + \Delta_{ij} - u_i)^2 / 2\sigma^2_{w,d_{w,ij}} \right\} \\
& \left. \times I(m_{ij} \leq 0)^{y_{ij}} I(m_{ij} > 0)^{1-y_{ij}} \right\} \\
& \times \frac{1}{(2\pi\sigma^2_{u,d_{u,i}})^{1/2}} \exp\left\{ -(u_i - \mu_{u,d_{u,i}})^2 / 2\sigma^2_{u,d_{u,i}} \right\} \\
& \left. \times \prod_{l=1}^{L_u} \pi_{u,l}^{I(d_{u,i}=l)} \right] \\
& \times \prod_{l=1}^{L_u} \frac{1}{(2\pi\theta)^{1/2}} \exp\left\{ -(\mu_{u,l} - \eta)^2 / 2\theta \right\} \\
& \times \prod_{l=1}^{L_u} \frac{\tau^\nu}{\gamma(\nu)} {\sigma^2_{u,l}}^{-\nu-1} \exp\left(\tau / \sigma^2_{u,l} \right) \\
& \times \prod_{l=1}^{L_u} \pi_{u,l}^{\alpha_l} \\
& \times \frac{1}{(2\pi|\Sigma|)^{1/2}} \exp\left\{ -(\beta^c - \mu_{\beta^c})^t \Sigma^{-1} (\beta^c - \mu_{\beta^c}) / 2 \right\},
\end{aligned}
$$

where X is an $IJ \times p$-matrix, m is an $IJ \times 1$-vector, β^c is a $p \times 1$-vector, u is an $IJ \times 1$-vector; π_w, μ_w, and σ^2_w are $L_w \times 1$-vectors; π_u, σ^2_u, and μ_u are $L_u \times 1$-vectors; and τ, ν, θ, η, μ_{β^c}, Σ are hyperparameters.

We now derive the full conditionals.

$$[d_{u,i}|\text{all else}] \propto \prod_{l=1}^{L_u} \pi_{u,l}^{I(d_{u,i}=l)} \frac{1}{\sigma_{u,d_{u,i}}} \exp\left\{-(u_i - \mu_{u,d_{u,i}})^2/2\sigma^2_{u,d_{u,i}}\right\}.$$

$$\begin{aligned}
[u_i|\text{all else}] &\propto \exp\left\{\frac{-1}{2\sigma^2_{u,d_{u,i}}}(u_i - \mu_{u,d_{u,i}})^2 - \sum_j \frac{1}{2\sigma^2_{w,d_{w,ij}}}(m_{ij} - \mu_{w,d_{w,ij}} + \Delta_{ij} - u_i)^2\right\} \\
&\propto \exp\left[\frac{-1}{2\sigma^2_{u,d_{u,i}}}(u_i^2 - 2\mu_{u,d_{u,i}}u_i) - \sum_j \frac{1}{2\sigma^2_{w,d_{w,ij}}}\left\{u_i^2 - 2(m_{ij} - \mu_{w,d_{w,ij}} + \Delta_{ij})u_i\right\}\right] \\
&\propto \exp\left\{\frac{-1}{2}u_i^2\left(\sum_j \frac{1}{\sigma^2_{w,d_{w,ij}}} + \frac{1}{\sigma^2_{u,d_{u,i}}}\right) + u_i\left(\sum_j \frac{m_{ij} - \mu_{w,d_{w,ij}} + \Delta_{ij}}{\sigma^2_{w,d_{w,ij}}} + \frac{\mu_{u,d_{u,i}}}{\sigma^2_{u,d_{u,i}}}\right)\right\}
\end{aligned}$$

$\Rightarrow u_i|\text{all else}$ is Normal with mean $\left(\sum_j \sigma^{-2}_{w,d_{w,ij}} + \sigma^{-2}_{u,d_{u,i}}\right)^{-1}\left(\sum_j \frac{m_{ij}-\mu_{w,d_{w,ij}}+\Delta_{ij}}{\sigma^2_{w,d_{w,ij}}} + \frac{\mu_{d_{u,i}}}{\sigma^2_{u,d_{u,i}}}\right)$ and variance $\left(\sum_j \sigma^{-2}_{w,d_{w,ij}} + \sigma^{-2}_{u,d_{u,i}}\right)^{-1}$.

$$[d_{w,ij}|\text{all else}] \propto \prod_{l=1}^{L_w} \pi_{w,l}^{I(d_{w,ij}=l)} \frac{1}{\sigma_{w,d_{w,ij}}} \exp\left\{-(m_{ij} - \mu_{w,d_{w,ij}} + \Delta_{ij} - u_i)^2/2\sigma^2_{w,d_{w,ij}}\right\}.$$

$$[m_{ij}|\text{all else}] \propto \frac{1}{\sigma_{w,d_{w,ij}}} \exp\left\{\frac{-1}{2\sigma^2_{w,d_{w,ij}}}(m_{ij} - \mu_{w,d_{w,ij}} + \Delta_{ij} - u_i)^2\right\} I(m_{ij} \le 0)^{y_{ij}} I(m_{ij} > 0)^{1-y_{ij}}$$

Implying

$$m_{ij}|\text{all else} \sim \frac{\Phi\left\{(m_{ij} - \mu_{w,d_{w,ij}} - u_i + \Delta_{ij})/\sigma_{w,d_{w,ij}}\right\}}{\Phi\left\{(-\mu_{w,d_{w,ij}} - u_i + \Delta_{ij})/\sigma_{w,d_{w,ij}}\right\}}$$

for $m_{ij} \le 0$, i.e. when $y_{ij} = 1$ and

$$m_{ij}|\text{all else} \sim \frac{\Phi\left\{(m_{ij} - \mu_{w,d_{w,ij}} - u_i + \Delta_{ij})/\sigma_{w,d_{w,ij}}\right\} - \Phi\left\{(-\mu_{w,d_{w,ij}} - u_i + \Delta_{ij})/\sigma_{w,d_{w,ij}}\right\}}{1 - \Phi\left\{(-\mu_{w,d_{w,ij}} - u_i + \Delta_{ij})/\sigma_{w,d_{w,ij}}\right\}}$$

for $m_{ij} \geq 0$, i.e. when $y_{ij} = 0$.

$$
\begin{aligned}
& [\beta^c|\text{all else}] \\
\propto \quad & \exp\{-(\beta^c - \mu_{\beta^c})^t \Sigma^{-1} (\beta^c - \mu_{\beta^c})/2\} \times \exp\left\{ \sum_{i,j} -(m_{ij} - \mu_{w,d_{w,ij}} + \Delta_{ij} - u_i)^2 / 2\sigma^2_{w,d_{w,ij}} \right\} \\
\propto \quad & \exp\left\{ \frac{-1}{2} [\beta^{c,t} \Sigma^{-1} \beta^c - 2(\Sigma^{-1}\mu_{\beta^c})^t \beta^c] - \frac{1}{2} [(X\beta^c)^t W^{-1} X \beta^c - 2(X^t W^{-1} \xi)^t \beta^c] \right\} \\
\propto \quad & \exp\left\{ \frac{-1}{2} \beta^{c,t} (\Sigma^{-1} + X^t W^{-1} X) \beta^c + \frac{2}{2} (\Sigma^{-1}\mu^c_\beta + X^t W^{-1} \xi)^t \beta^c \right\} \\
\Rightarrow \beta^c|\text{all else} \quad \sim \quad & MVN\left((\Sigma^{-1} + X^t W^{-1} X)^{-1} (\Sigma^{-1}\mu_{\beta^c} + X^t W^{-1} \xi), (\Sigma^{-1} + X^t W^{-1} X)^{-1} \right).
\end{aligned}
$$

where X is the design matrix, W is a diagonal matrix of the $\sigma^2_{w,d_{w,ij}}$ and ξ is a vector with elements $\mu_{w,d_{w,ij}} + u_i - m_{ij}$.

$$
\begin{aligned}
& [\sigma^2_{u,l}|\text{all else}] \\
\propto \quad & \prod_i \left[\frac{1}{\sigma_{u,l}} \exp\left\{ -(u_i - \mu_{u,d_{u,l}})^2 / 2\sigma^2_{u,l} \right\} \right]^{I(d_{u,i}=l)} \frac{\tau^\nu}{\gamma(\nu)} \sigma_{u,l}^{2-\nu-1} \exp\left(\frac{-\tau}{\sigma^2_{u,l}} \right) \\
\propto \quad & \sigma_{u,l}^{2-\nu-\sum_i I(d_{u,i}=l)/2-1} \exp\left\{ \frac{-\tau - \sum_i (u_i - \mu_{u,d_{u,l}})^2 I(d_{u,i}=l)/2}{\sigma^2_{u,l}} \right\}
\end{aligned}
$$

$\Rightarrow \sigma^2_{u,l}|$ all else is an inverse gamma with shape parameter $\nu + \sum_i I(d_{u,i} = l)/2$ and rate parameter $\tau + \sum_i I(d_{u,i} = l)(u_i - \mu_{u,l})^2/2$.

$$
\begin{aligned}
& [\mu_{u,l}|\text{all else}] \\
\propto \quad & \frac{1}{\theta^{1/2}} \exp\left\{ -(\mu_{u,d_{u,l}} - \eta)^2 / 2\theta \right\} \prod_i \left[I(d_{u,i} = l) \frac{1}{\sigma_{u,d_{u,i}}} \exp\left\{ (u_i - \mu_{u,d_{u,i}})^2 / 2\sigma^2_{u,d_{u,i}} \right\} \right] \\
\propto \quad & \exp\left\{ \frac{-1}{2} \mu^2_{u,d_{u,l}} \frac{1}{\theta} + \frac{\eta}{\theta} \mu_{u,d_{u,l}} \right\} \exp\left\{ \frac{-1}{2} \mu^2_{u,d_{u,l}} \sum_i \frac{I(d_{u,i} = l)}{\sigma^2_{u,d_{u,i}}} + \mu_{u,d_{u,l}} \sum_i I(d_{u,i} = l) \frac{u_i}{\sigma^2_{u,d_{u,i}}} \right\}.
\end{aligned}
$$

$\Rightarrow \mu_{u,l}|$ all else is Normal with mean $\left(\sum_i I(d_{u,i}=l)\frac{1}{\sigma^2_{u,d_{u,i}}}+\frac{1}{\theta}\right)^{-1}\left(\sum_i I(d_{u,i}=l)\frac{u_i}{\sigma^2_{u,d_{u,i}}}+\frac{\eta}{\theta}\right)$ and variance $\left(\sum_i I(d_{u,i}=l)\frac{1}{\sigma^2_{u,d_{u,i}}}+\frac{1}{\theta}\right)^{-1}$.

$$\begin{aligned}
[\pi_{u,l}|\text{all else}] &\propto \prod_i \left\{\pi_{u,l}^{I(d_{u,i}=l)}\right\}\pi_{u,l}^{\alpha_l} \\
&= \pi_{u,l}^{\sum_i I(d_{u,i}=l)+\alpha_l}
\end{aligned}$$

$\Rightarrow \pi_{u,l}|$ all else is Dirichlet with shape parameter $\alpha_l+\sum_i I(d_{u,i}=l)$.
