

Derivation of the full conditionals for Bayesian analysis

In the following, we derive the straightforward calculations of the full conditionals for Bayesian analysis. For the Bayesian model specified in the paper, the likelihood is given by:

$$\begin{aligned}
& f(y, d_w, m, u, d_u, \mu_u, \sigma_{u,d_u}^2, \pi_u, \beta^c) \\
&= \prod_{i=1}^I \prod_{j=1}^J f(y_{ij}, d_{w,ij}, m_{ij}, u_i, d_{u,i}, \mu_u, \sigma_u^2, \pi_u, \beta^c) \\
&= \prod_{i=1}^I \left[\prod_{j=1}^J \left\{ \right. \right. \\
&\quad \prod_{l=1}^{L_w} \pi_{w,l}^{I(d_{w,ij}=l)} \\
&\quad \times \frac{1}{(2\pi\sigma_{w,d_{w,ij}}^2)^{1/2}} \exp \left\{ -(m_{ij} - \mu_{w,d_{w,ij}} + \Delta_{ij} - u_i)^2 / 2\sigma_{w,d_{w,ij}}^2 \right\} \\
&\quad \times I(m_{ij} \leq 0)^{y_{ij}} I(m_{ij} > 0)^{1-y_{ij}} \left. \right\} \\
&\quad \times \frac{1}{(2\pi\sigma_{u,d_{u,i}}^2)^{1/2}} \exp \left\{ -(u_i - \mu_{u,d_{u,i}})^2 / 2\sigma_{u,d_{u,i}}^2 \right\} \\
&\quad \times \prod_{l=1}^{L_u} \pi_{u,l}^{I(d_{u,i}=l)} \left. \right] \\
&\quad \times \prod_{l=1}^{L_u} \frac{1}{(2\pi\theta)^{1/2}} \exp \left\{ -(\mu_{u,l} - \eta)^2 / 2\theta \right\} \\
&\quad \times \prod_{l=1}^{L_u} \frac{\tau^\nu}{\gamma(\nu)} \sigma_{u,l}^2{}^{-\nu-1} \exp(\tau/\sigma_{u,l}^2) \\
&\quad \times \prod_{l=1}^{L_u} \pi_{u,l}^{\alpha_l} \\
&\quad \times \frac{1}{(2\pi|\Sigma|)^{1/2}} \exp \left\{ -(\beta^c - \mu_{\beta^c})^t \Sigma^{-1} (\beta^c - \mu_{\beta^c}) / 2 \right\},
\end{aligned}$$

where X is an $IJ \times p$ -matrix, m is an $IJ \times 1$ -vector, β^c is a $p \times 1$ -vector, u is an $IJ \times 1$ -vector; π_w , μ_w , and σ_w^2 are $L_w \times 1$ -vectors; π_u , σ_u^2 , and μ_u are $L_u \times 1$ -vectors; and τ , ν , θ , η , μ_{β^c} , Σ are hyperparameters.

We now derive the full conditionals.

$$[d_{u,i}|\text{all else}] \propto \prod_{l=1}^{L_u} \pi_{u,l}^{I(d_{u,i}=l)} \frac{1}{\sigma_{u,d_{u,i}}} \exp \left\{ -(u_i - \mu_{u,d_{u,i}})^2 / 2\sigma_{u,d_{u,i}}^2 \right\}.$$

$$\begin{aligned} [u_i|\text{all else}] &\propto \exp \left\{ \frac{-1}{2\sigma_{u,d_{u,i}}^2} (u_i - \mu_{u,d_{u,i}})^2 - \sum_j \frac{1}{2\sigma_{w,d_{w,ij}}^2} (m_{ij} - \mu_{w,d_{w,ij}} + \Delta_{ij} - u_i)^2 \right\} \\ &\propto \exp \left[\frac{-1}{2\sigma_{u,d_{u,i}}^2} (u_i^2 - 2\mu_{u,d_{u,i}} u_i) - \sum_j \frac{1}{2\sigma_{w,d_{w,ij}}^2} \{ u_i^2 - 2(m_{ij} - \mu_{w,d_{w,ij}} + \Delta_{ij}) u_i \} \right] \\ &\propto \exp \left\{ \frac{-1}{2} u_i^2 \left(\sum_j \frac{1}{\sigma_{w,d_{w,ij}}^2} + \frac{1}{\sigma_{u,d_{u,i}}^2} \right) + u_i \left(\sum_j \frac{m_{ij} - \mu_{w,d_{w,ij}} + \Delta_{ij}}{\sigma_{w,d_{w,ij}}^2} + \frac{\mu_{u,d_{u,i}}}{\sigma_{u,d_{u,i}}^2} \right) \right\} \end{aligned}$$

$\Rightarrow u_i|\text{all else}$ is Normal with mean $\left(\sum_j \sigma_{w,d_{w,ij}}^{-2} + \sigma_{u,d_{u,i}}^{-2} \right)^{-1} \left(\sum_j \frac{m_{ij} - \mu_{w,d_{w,ij}} + \Delta_{ij}}{\sigma_{w,d_{w,ij}}^2} + \frac{\mu_{u,d_{u,i}}}{\sigma_{u,d_{u,i}}^2} \right)$
and variance $\left(\sum_j \sigma_{w,d_{w,ij}}^{-2} + \sigma_{u,d_{u,i}}^{-2} \right)^{-1}$.

$$[d_{w,ij}|\text{all else}] \propto \prod_{l=1}^{L_w} \pi_{w,l}^{I(d_{w,ij}=l)} \frac{1}{\sigma_{w,d_{w,ij}}} \exp \left\{ -(m_{ij} - \mu_{w,d_{w,ij}} + \Delta_{ij} - u_i)^2 / 2\sigma_{w,d_{w,ij}}^2 \right\}.$$

$$[m_{ij}|\text{all else}] \propto \frac{1}{\sigma_{w,d_{w,ij}}} \exp \left\{ \frac{-1}{2\sigma_{w,d_{w,ij}}^2} (m_{ij} - \mu_{w,d_{w,ij}} + \Delta_{ij} - u_i)^2 \right\} I(m_{ij} \leq 0)^{y_{ij}} I(m_{ij} > 0)^{1-y_{ij}}$$

Implying

$$m_{ij}|\text{all else} \sim \frac{\Phi \left\{ (m_{ij} - \mu_{w,d_{w,ij}} - u_i + \Delta_{ij}) / \sigma_{w,d_{w,ij}} \right\}}{\Phi \left\{ (-\mu_{w,d_{w,ij}} - u_i + \Delta_{ij}) / \sigma_{w,d_{w,ij}} \right\}}$$

for $m_{ij} \leq 0$, i.e. when $y_{ij} = 1$ and

$$m_{ij}|\text{all else} \sim \frac{\Phi \left\{ (m_{ij} - \mu_{w,d_{w,ij}} - u_i + \Delta_{ij}) / \sigma_{w,d_{w,ij}} \right\} - \Phi \left\{ (-\mu_{w,d_{w,ij}} - u_i + \Delta_{ij}) / \sigma_{w,d_{w,ij}} \right\}}{1 - \Phi \left\{ (-\mu_{w,d_{w,ij}} - u_i + \Delta_{ij}) / \sigma_{w,d_{w,ij}} \right\}}$$

for $m_{ij} \geq 0$, i.e. when $y_{ij} = 0$.

$$\begin{aligned}
& [\beta^c | \text{all else}] \\
& \propto \exp\{-(\beta^c - \mu_{\beta^c})^t \Sigma^{-1} (\beta^c - \mu_{\beta^c}) / 2\} \times \exp\left\{\sum_{i,j} -(m_{ij} - \mu_{w,d_w,ij} + \Delta_{ij} - u_i)^2 / 2\sigma_{w,d_w,ij}^2\right\} \\
& \propto \exp\left\{\frac{-1}{2}[\beta^{c,t} \Sigma^{-1} \beta^c - 2(\Sigma^{-1} \mu_{\beta^c})^t \beta^c] - \frac{1}{2}[(X \beta^c)^t W^{-1} X \beta^c - 2(X^t W^{-1} \xi)^t \beta^c]\right\} \\
& \propto \exp\left\{\frac{-1}{2} \beta^{c,t} (\Sigma^{-1} + X^t W^{-1} X) \beta^c + \frac{2}{2} (\Sigma^{-1} \mu_{\beta^c}^c + X^t W^{-1} \xi)^t \beta^c\right\} \\
\Rightarrow \beta^c | \text{all else} & \sim MVN((\Sigma^{-1} + X^t W^{-1} X)^{-1} (\Sigma^{-1} \mu_{\beta^c}^c + X^t W^{-1} \xi), (\Sigma^{-1} + X^t W^{-1} X)^{-1}).
\end{aligned}$$

where X is the design matrix, W is a diagonal matrix of the $\sigma_{w,d_w,ij}^2$ and ξ is a vector with elements $\mu_{w,d_w,ij} + u_i - m_{ij}$.

$$\begin{aligned}
& [\sigma_{u,l}^2 | \text{all else}] \\
& \propto \prod_i \left[\frac{1}{\sigma_{u,l}} \exp\left\{-\frac{(u_i - \mu_{u,d_u,l})^2}{2\sigma_{u,l}^2}\right\} \right]^{I(d_{u,i}=l)} \frac{\tau^\nu}{\gamma(\nu)} \sigma_{u,l}^{2-\nu-1} \exp\left(\frac{-\tau}{\sigma_{u,l}^2}\right) \\
& \propto \sigma_{u,l}^{2-\nu-\sum_i I(d_{u,i}=l)/2-1} \exp\left\{\frac{-\tau - \sum_i (u_i - \mu_{u,d_u,l})^2 I(d_{u,i}=l)/2}{\sigma_{u,l}^2}\right\}
\end{aligned}$$

$\Rightarrow \sigma_{u,l}^2 | \text{all else}$ is an inverse gamma with shape parameter $\nu + \sum_i I(d_{u,i}=l)/2$ and rate parameter $\tau + \sum_i I(d_{u,i}=l)(u_i - \mu_{u,l})^2/2$.

$$\begin{aligned}
& [\mu_{u,l} | \text{all else}] \\
& \propto \frac{1}{\theta^{1/2}} \exp\left\{-\frac{(\mu_{u,d_u,l} - \eta)^2}{2\theta}\right\} \prod_i \left[I(d_{u,i}=l) \frac{1}{\sigma_{u,d_u,i}} \exp\left\{\frac{(u_i - \mu_{u,d_u,i})^2}{2\sigma_{u,d_u,i}^2}\right\} \right] \\
& \propto \exp\left\{\frac{-1}{2} \mu_{u,d_u,l}^2 \frac{1}{\theta} + \frac{\eta}{\theta} \mu_{u,d_u,l}\right\} \exp\left\{\frac{-1}{2} \mu_{u,d_u,l}^2 \sum_i \frac{I(d_{u,i}=l)}{\sigma_{u,d_u,i}^2} + \mu_{u,d_u,l} \sum_i I(d_{u,i}=l) \frac{u_i}{\sigma_{u,d_u,i}^2}\right\}.
\end{aligned}$$

$\Rightarrow \mu_{u,l} | \text{all else}$ is Normal with mean $\left(\sum_i I(d_{u,i} = l) \frac{1}{\sigma_{u,d_{u,i}}^2} + \frac{1}{\theta} \right)^{-1} \left(\sum_i I(d_{u,i} = l) \frac{u_i}{\sigma_{u,d_{u,i}}^2} + \frac{\eta}{\theta} \right)$
 and variance $\left(\sum_i I(d_{u,i} = l) \frac{1}{\sigma_{u,d_{u,i}}^2} + \frac{1}{\theta} \right)^{-1}$.

$$\begin{aligned}
 [\pi_{u,l} | \text{all else}] &\propto \prod_i \left\{ \pi_{u,l}^{I(d_{u,i}=l)} \right\} \pi_{u,l}^{\alpha_l} \\
 &= \pi_{u,l}^{\sum_i I(d_{u,i}=l) + \alpha_l}
 \end{aligned}$$

$\Rightarrow \pi_{u,l} | \text{all else}$ is Dirichlet with shape parameter $\alpha_l + \sum_i I(d_{u,i} = l)$.
