Advanced Theory

Survival Analysis 2005

Problems for February 22, 2005

1) $h_F(t|F) = \exp(b_0 + b_1 F)$

Find the MLE for b_0 and b_1 and for variance($\widehat{b_0}$), variance($\widehat{b_1}$), and covariance($\widehat{b_0}$,$\widehat{b_1}$).

We assume that $T \sim \exp(\lambda)$. Let $B = \{b_0, b_1\}$. Let $\sum_j^{n_1}$ represent summing over all those in whom $F_j = 0$, and $\sum_j^{n_2}$ represent summing over the remaining group.

$$L(B) \propto \prod_i^n exp(b_0 + b_1 F_i) \exp(-\exp(b_0 + F_i b_1) t_i)$$

$$\log\ L(B) \propto \sum_i^n (b_0 + b_1 F_i) - \exp(b_0 + F_i b_1) t_i$$

$$\frac{\partial}{\partial b_1} \log\ L(B) = \sum_i^n F_i - F_i \exp(b_0 + F_i b_1) t_i$$

Solve for the value that sets the above equation to 0.

$$0 = \sum_i^n F_i - F_i t_i \exp(b_0 + F_i b_1)$$

$$0 = \sum_i^{n_2} 1 - t_i \exp(b_0 + b_1)$$

$$0 = n_2 - \exp(b_0 + b_1) \sum_i^{n_2} t_i$$

$$\log(n_2) = b_0 + b_1 + \log(\sum_i^{n_2} t_i)$$

$$b_1 = \log(n_2 / \sum_i^{n_2} t_i) - b_0$$

$$\frac{\partial}{\partial b_0} \log \; L(B) = \sum_i^n (1 - t_i \exp(b_0 + F_i b_1))$$

Set to 0.

$$0 = \sum_i^n (1 - t_i \exp(b_0 + F_i b_1))$$

$$0 = n - \sum_j^{n_1} t_i \exp(b_0) - \sum_i^{n_2} t_i \exp(b_0 + b_1)$$

Substitute for b_1

$$0 = n - \sum_j^{n_1} t_i \exp(b_0) - \sum_i^{n_2} t_i \exp(\log(n_2 / \sum_i^{n_2} t_i))$$

$$0 = n - \sum_j^{n_1} t_i \exp(b_0) - n_2$$

$$\log n_1 = log(\sum_j^{n_1} t_i) + b_0$$

$$\widehat{b}_0 = \log(n_1 / log(\sum_j^{n_1} t_i))$$

Hence,

$$\widehat{b}_1 = \log(n_2 / \sum_i^{n_2} t_i) - \log(n_1 / log(\sum_j^{n_1} t_i))$$

To find the variance:

Let $b = \{b_0, b_1\}$. By MLE theory, we know that,

$$\sqrt{n}(\hat{b} - b) \xrightarrow{d} MVN(0, I^{-1})$$

where

$$I^{-1} = -E \begin{bmatrix} \frac{\partial^2}{\partial b_0^2} logL(b|t_1) & \frac{\partial^2}{\partial b_0 b_1} logL(b|t_1) \\ \frac{\partial^2}{\partial b_0 b_1} logL(b|t_1) & \frac{\partial^2}{\partial b_1^2} logL(b|t_1) \end{bmatrix}^{-1}$$

$$\frac{\partial^2}{\partial b_1^2}\log\ L(B|t_i) = -F_1\exp(b_0+F_1b_1)t_i$$

$$-E[-F_1\exp(b_0+F_1b_1)T_1]$$

$$=\{\exp(b_0+b_1)E[T_1|F_1=1]P(F1=1)\}$$

$$=\{\exp(b_0+b_1)/\exp(b_0+b_1)P(F_1=1)\}$$

$$=P(F_1=1)$$

$$\frac{\partial^2}{\partial b_0^2}\log\ L(B) = -\exp(b_0+F_1b_1)t_1$$

$$-E[-\exp(b_0+F_1b_1)t_1]$$

$$=\{\exp(b_0+b_1)E[t_1|F=1]P(F=1)+\exp(b_0)E[t_1|F=0]P(F=0)\}$$

$$=\{P(F=1)+P(F=0)\}=1$$

$$\frac{\partial^2}{\partial b_0\partial b_1}\log\ L(B) = -F_1\exp(b_0+F_1b_1)t_1$$

$$-E[-F_1\exp(b_0+F_1b_1)T_1]$$

$$=P(F_1=1)$$

Hence,

$$I^{-1}=\begin{bmatrix} 1 & P(F_1=1) \\ P(F_1=1) & P(F_1=1) \end{bmatrix}^{-1}$$

On 3/1/2005, we will use other methods to check our results.

2) Prove:

If Y has a Weibull distribution then,

$$E[Y^r]=\lambda^{-r/\gamma}\Gamma(1+r/\gamma)$$

For $\lambda=1/3$, $\gamma=2$, $a=2$, and $b=4$, find E[Y], Variance(Y), and $P(Y\geq b\mid Y\geq a)$

a)

$$E[Y^r] = \lambda^{-r/\gamma}\Gamma(1 + r/\gamma)$$
$$= \int t^r \lambda\gamma t^{\gamma-1} \exp(-\lambda t^\gamma) dt$$

Let's do a change of variable transformation. Let x=λt^γ. Then, $t = (x/\lambda)^{1/\gamma}$ and $\frac{d}{dt}t = 1/\gamma(x/\lambda)^{1/\gamma-1}$. Hence,

$$E[Y^r] = \lambda^{-r/\gamma}\Gamma(1 + r/\gamma)$$
$$= \int (x/\lambda)^{r/\gamma}\gamma\lambda(x/\lambda)^{(\gamma-1)/\gamma} \exp(-x)\frac{1}{\gamma\lambda}(x/\lambda)^{(1-\gamma)/\gamma}dx$$
$$= \int (x/\lambda)^{r/\gamma} \exp(-x)dx$$
$$= \lambda^{-r/\gamma} \int x^{(1+r/\gamma)-1} \exp(-x)dx$$
$$= \lambda^{-r/\gamma}\Gamma(1 + r/\gamma)$$

b) By substitution, we see that,

$$E[Y] = \sqrt{3}\Gamma(3/2) = 1.5$$
$$E[Y^2] = 3\Gamma(2) = 3$$
$$Var(Y) = 3 - 1.5^2 = .75$$

c) Use Bayes' theorem to see,

$$P(Y \geq 4|Y \geq 4) = P(Y \geq 4)/P(Y \geq 2)$$

$$P(Y \geq 2) = \int_2^\infty 2/3t \exp(-t^2/3)dt$$
$$= \exp(-t^2/3)|_2^\infty$$
$$= 1 - \exp(-4/3)$$

$$P(Y \geq 4) = \int_4^\infty 2/3t \exp(-t^2/3)dt$$

$$= \exp(-t^2/3)|_4^\infty$$

$$= 1 - \exp(-16/3)$$

$$P(Y \geq 4 | Y \geq 4) = [1 - \exp(-4/3)]/[1 - \exp(-2/3)] = 1.51$$