Advanced Theory

Survival Analysis 2005

Problem for February 15, 2005

$t_1, t_2, ...t_m$ are survival times and $S(t) = \exp(-\lambda t)$.

Show:

1) $f(t) = \lambda \exp(-\lambda t)$

Proof:

$$f(t) = -\frac{d}{dt} S(t) = \lambda \exp(-\lambda t)$$

2) The maximum likelihood estimate of λ is

$$\widehat{\lambda} = \left(\sum_{i=1}^{d} t_i / d \right)^{-1}$$

where d= number of complete observations.

Proof:

$$L(\lambda) \propto \prod_{i}^{d} \lambda \exp(-\lambda t_i)$$

$$\log\ L(\lambda) = \sum_{i}^{d} \{\log(\lambda) - \lambda t_i\}$$

$$\frac{d}{d\lambda} log L(\lambda) = \sum_{i}^{d} \left\{ \frac{1}{\lambda} - t_i \right\}$$

Solve for λ:

$$\sum_{i}^{d} \left\{ \frac{1}{\lambda} - t_i \right\} = 0$$

Solve for λ:

$$\frac{d}{\lambda} = \sum_{i}^{d} t_i$$

$$\widehat{\lambda} = \frac{d}{\sum_i^d t_i}$$

3) Variance of $\widehat{\lambda}$ is estimated by,

$$\text{variance}(\widehat{\lambda}) = \widehat{\lambda}^2/d$$

Proof:

Let $L(\lambda \mid t_i)$ be the log likelihood given one observed value, t_i.

$$-E[\frac{d^2}{d\lambda^2} \log L(\lambda \mid t_i)]^{-1}$$

$$-E[\frac{d^2}{d\lambda^2} (log(\lambda) - \lambda t_i)]^{-1}$$

$$= -E[-\frac{1}{\lambda}^2]^{-1}$$

$$= \lambda^2$$

Hence, by maximum likelihood theory,

$$\sqrt{d}(\widehat{\lambda} - \lambda) \xrightarrow{d} N(0, \lambda^2)$$

Thus, a finite sample estimate of the variance of $\widehat{\lambda}$ is $\frac{\widehat{\lambda}^2}{d}$.