This is an independent take-home assignment. That is, you are not allowed to collaborate (communicate) with each other, other students, faculty, or anyone else about the assignment. You are not allowed to look at solutions to these problems. Please show all your work as credit will not be given for undocumented answers. Questions of clarifications should be addressed directly to me, Zebin or Elizabeth. If there is any ambiguity about the above instructions, please contact me. On your assignment, please sign the statement, “I have neither given nor received aid on this homework assignment.” Good luck!

1. Let $X_1, \ldots, X_n$ be i.i.d. random variables with density

$$f(x; \alpha) = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2} \{(x(1-x))^{\alpha-1} \} x \in [0,1], \alpha > 0$$

- What is the mean and variance of a random observation from this distribution?
- Derive a method of moment estimator for $\alpha$ based on $n$ observations.

2. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be i.i.d. random vectors with joint density

$$f(x, y; \mu_x, \mu_y, \rho) = \frac{1}{2\pi \sqrt{1-\rho^2}} \exp \left\{ -\frac{(x - \mu_x)^2 - 2\rho(x - \mu_x)(y - \mu_y) + (y - \mu_y)^2}{2(1-\rho^2)} \right\}$$

where $(x, y) \in \mathbb{R} \times \mathbb{R}$ and $(\mu_x, \mu_y, \rho) \in \mathbb{R} \times \mathbb{R} \times (-1,1)$.

- Derive method of moments estimators for $\mu_x$, $\mu_y$ and $\rho$.
- Derive its limiting distribution. The asymptotic variance will depend on $\mu_x$, $\mu_y$, $\rho$ and various cross moments of $X$ and $Y$ - write an expression that depends on these quantities.
- Propose estimators for the standard error of the MOM estimators.

3. Suppose you have data from $K$ studies. Let $Y_{k} = (Y_{1,k}, \ldots, Y_{n_k,k})$ represent $n_k$ observations from study $k$, $k = 1, \ldots, K$. Let

$$Y_{i,k} = \mu + \theta_k + \epsilon_{i,k} \ i = 1, \ldots, n_k, k = 1, \ldots, K$$

where $\theta_1, \ldots, \theta_K$ are independent, $\theta_k \sim N(0, \tau^2)$, $\epsilon_{i,k} \overset{i.i.d.}{\sim} N(0, \sigma_k^2)$, $\mu$, $\tau^2$, and $\sigma_1^2, \ldots, \sigma_K^2$ are fixed parameters. We are assuming that, given $\theta_1, \ldots, \theta_K, Y_{1,k}, \ldots, Y_{n_k,k}$ are i.i.d. $N(\mu + \theta_k, \sigma_k^2)$ and observations are independent across studies.

- What is the marginal distribution of $Y_{i,k}$?
- Propose a method of moments estimator for $\sigma_k^2$. Denote these estimators as $\hat{\tau}_k^2$.
- Let $w_1, \ldots, w_K$ be study-specific positive fixed weights. Let $\overline{Y}_k$ be the mean of the observation in study $k$. Let

$$\overline{Y} = \frac{\sum_{k=1}^{K} w_k \overline{Y}_k}{\sum_{k=1}^{K} w_k}$$

What is the expected value of $\sum_{k=1}^{K} w_k(\overline{Y}_k - \overline{Y})^2$?
- Use the above results to construct a method of moments estimator of $\tau^2$.  

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