1. Prior: $p(\theta) \propto \theta^3(1-\theta)^3$,

   Likelihood: $p(y|\theta) \propto \binom{10}{0}(1-\theta)^10 + \binom{10}{1}\theta(1-\theta)^9 + \binom{10}{2}\theta^2(1-\theta)^8$
   $= (1-\theta)^10 + 10\theta(1-\theta)^9 + 45\theta^2(1-\theta)^8$.

   Then, the exact posterior density (up to a proportionality constant) for θ is
   $p(\theta|y) \propto theta^3(1-\theta)^{13} + 10\theta^4(1-\theta)^{12} + 45\theta^5(1-\theta)^{11}$.

   In order to calculate the posterior mean, variance and 95% interval, we adopt the simulation-based inference approach. By realizing that the posterior results from a *mixture* of three Beta distributions, we have to find the weights π_1, π_2, π_3, $\sum \pi_i = 1$, such that

   $$p(\theta|y) = \pi_1 Beta(4,14) + \pi_2 Beta(5,13) + \pi_3 Beta(6,12) \propto \theta^3(1-\theta)^{13} + 10\theta^4(1-\theta)^{12} + 45\theta^5(1-\theta)^{11}.$$

   Since $Beta(\alpha,\beta) = c\,\theta^{\alpha-1}(1-\theta)^{\beta-1}$, where $c = \frac{1}{Beta(\alpha,\beta)}$ is the normalizing constant,

   $$\pi_1 = \frac{\frac{1}{c_1}}{\sum \pi_i}, \quad \pi_2 = \frac{\frac{10}{c_2}}{\sum \pi_i}, \quad \pi_3 = \frac{\frac{45}{c_3}}{\sum \pi_i},$$

   are the weights we look for. In fact,

   $$\pi_1 Beta(4,14) + \pi_2 Beta(5,13) + \pi_3 Beta(6,12) = \frac{1}{\sum \pi_i} \cdot \left(\theta^3(1-\theta)^{13} + 10\theta^4(1-\theta)^{12} + 45\theta^5(1-\theta)^{11}\right).$$

   To draw $p(\theta|y)$ and calculate posterior mean, variance and 95% confidence interval, give the following commands at R console:

```
>pi.sum_sum(1*beta(4,14)+10*beta(5,13)+45*beta(6,12))
>pi1_ 1*beta(4,14)/pisum
>pi1
[1] 0.1015625
>pi2_ 10*beta(5,13)/pisum
>pi2
[1] 0.3125
>pi3_45*beta(6,12)/pisum
>pi3
[1] 0.5859375
>m1_rbeta(10156,4,14)
>m2_rbeta(31250,5,13)
>m3_rbeta(58594,6,12)
>mean(c(m1,m2,m3))
[1] 0.3051950 # otherwise, to calculate the posterior mean:
> pi1*4/18+pi2*5/18+pi3*6/18
[1] 0.3046875
>ci_quantile( c(m1,m2,m3),c(.025,.975))
>ci
     2.5%     97.5%
0.1095699 0.5413070
>v_var(c(m1,m2,m3))
>v
[1] 0.01253171
> theta_seq(0,1,.001)
> dens_theta^3*(1-theta)^13+10*theta^4*(1-theta)^12+45*theta^5*(1-theta)^11
> plot(theta,dens,ylim=c(0,max(dens)),type="l",xlab="theta",ylab="density",
```

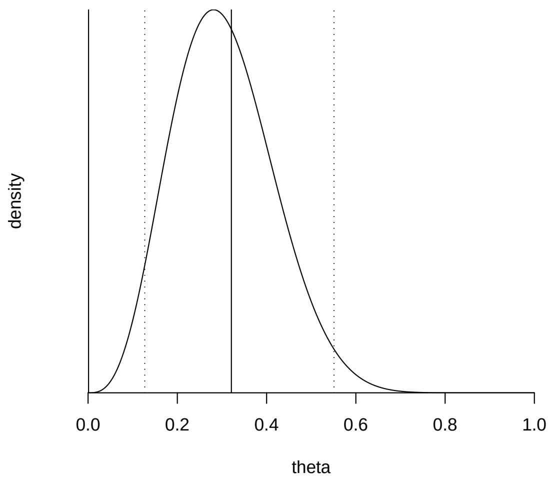
```
> xaxs="i",yaxs="i",yaxt="n",bty="n",cex=2)
> abline(v=m,lty=2,lwd=2)
> abline(v=ci,lty=3)
```

An alternative way to generate a mixture distribution is coded as:

```
mixture.r_function(m,m.pi,n)
{
x_seq(0,m,1)
champ_sample(x,size=n,prob=m.pi,replace=T)
y_c()
for(i in 1:n)
 {y_c(y,switch(champ[i]+1,rbeta(1,4,14),rbeta(1,5,13),rbeta(1,6,12)))}
return(y)
}
```

Then:

```
>mixt_ mixture.r(2,c(pi1,pi2,pi3),10000)
>mean(mixt)
[1] 0.3046123
```



2 2a

$$\begin{aligned} Pr(y=k) &= \int_0^1 Pr(y=k|\theta)d\theta \\ &= \int_0^1 \binom{n}{k}\theta^k(1-\theta)^{n-k}d\theta \\ &= \binom{n}{k}\frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+2)} \qquad (1) \\ &= \frac{n!}{(n-k)!k!}\frac{k!(n-k)!}{(n+1)!} \qquad (2) \\ &= \frac{1}{n+1} \end{aligned}$$

(1) holds as $\int_0^1 \theta^k(1-\theta)^{n-k}d\theta = \frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+2)}$
(2) holds as $\Gamma(x) = (x-1)!$

2b Posterior mean is $\frac{\alpha+y}{\alpha+\beta+n}$. To show that it lies between $\frac{\alpha}{\alpha+\beta}$ and $\frac{y}{n}$, we will write it as $\frac{\alpha+y}{\alpha+\beta+n} = \lambda \frac{\alpha}{\alpha+\beta} + (1-\lambda)\frac{y}{n}$, and show that $0 < \lambda < 1$. To do this, solve for λ:

$$\begin{aligned}
\frac{\alpha+y}{\alpha+\beta+n} &= \frac{y}{n} + \lambda\left(\frac{\alpha}{\alpha+\beta} - \frac{y}{n}\right) \\
\frac{\alpha+y}{\alpha+\beta+n} - \frac{y}{n} &= \lambda\left(\frac{\alpha}{\alpha+\beta} - \frac{y}{n}\right) \\
\frac{n\alpha - \alpha y - \beta y}{(\alpha+\beta+n)n} &= \lambda(\frac{n\alpha - \alpha y - \beta y}{(\alpha+\beta)n} \\
\lambda &= \frac{\alpha+\beta}{\alpha+\beta+n},
\end{aligned}$$

which is always between 0 and 1. So the posterior mean is a weighted average of the prior mean and the data.