Solutions of Homework 2

1. • $N(y|\mu,\sigma^2)$ belongs to the 2-parameter exponential family. In fact,

$$N(y|\mu,\sigma^2) = \frac{1}{\sqrt{(2\pi)}}\,\frac{1}{\sigma}\,exp\left\{-\frac{\mu^2}{2\sigma^2}\right\}\ exp\left\{\frac{\mu}{\sigma^2}y\ -\frac{1}{2}\frac{1}{\sigma^2}y^2\right\} \tag{1}$$

If $y_i \sim N(\mu,\sigma^2)$ iid for $i=1,\ldots,n$ and let $\boldsymbol{y}=(y_1,y_2,\ldots,y_n)$ then

$$\begin{aligned} p(\boldsymbol{y}\mid\mu,\sigma^2) &= \sigma^{-n}\exp\left[\left(-\tfrac{\mu^2}{2\sigma^2},\tfrac{\mu}{\sigma^2},-\tfrac{1}{2\sigma^2}\right)\begin{pmatrix} n \\ \sum y_i \\ \sum y_i^2\end{pmatrix}\right] \\ &= g(\boldsymbol{\theta})^n\exp\left(\phi(\boldsymbol{\theta})^t t(\boldsymbol{y})\right)\end{aligned}$$

where $\boldsymbol{\theta}=(\mu,\sigma^2)$.

• In the exponential form (1)

$$\begin{gathered} \mathbf{c}=(1,-1/2) \\ \boldsymbol{\phi}(\boldsymbol{\theta})=(\tfrac{\mu}{\sigma^2},\tfrac{1}{\sigma^2}) \\ \mathbf{h}(y)=(y,y^2) \\ \mathbf{t}_n(y_1,\ldots,y_n)=(n,\textstyle\sum_{i=1}^n y_i,\sum_{i=1}^n y_i^2), \end{gathered} \tag{2}$$

$n=1,2,\ldots$ is the sequence of sufficient statistics.

• Conjugate prior

$$\begin{aligned} p(\boldsymbol{\theta}) &\propto \sigma^{-\nu_0}\exp\left(\phi(\boldsymbol{\theta})\boldsymbol{\nu}\right) \\ &\propto (\sigma^2)^{-\frac{\nu_0-1}{2}}\exp\left(-\tfrac{\nu_3}{2\sigma^2}\right)\times(\sigma^2)^{-\frac{1}{2}}\exp\left(-\tfrac{\mu^2}{2\sigma^2}\nu_1+\tfrac{\mu}{\sigma^2}\nu_2\right) \\ &\propto \underbrace{(\sigma^2)^{-(\frac{\nu_0}{2}-1)}\exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right)}_{IG(\frac{\nu_0}{2},\frac{\nu_0\sigma_0^2}{2})}\times\underbrace{(\sigma^2)^{-\frac{1}{2}}\exp\left(-\frac{\kappa_0}{2\sigma^2}(\mu-\mu_0)^2\right)}_{N(\mu_0,\frac{\sigma^2}{\kappa_0})}\end{aligned}$$

where

$$\begin{aligned}\nu_1 &= \kappa_0 \\ \nu_2 &= \kappa_0\mu_0 \\ \nu_3 &= \nu_0\sigma_0^2\end{aligned}$$

$$p(\boldsymbol{\theta}\mid\boldsymbol{y}) \quad\propto\quad IG(\tfrac{\nu_n}{2},\tfrac{\nu_n\sigma^2}{2})N(\mu_n,\tfrac{\sigma_n^2}{\kappa_n})$$

where

$$\begin{aligned}\nu_n &= \nu_0+n \\ \nu_n\sigma_n^2 &= \nu_0\sigma_0^2+(n-1)s^2+\tfrac{\kappa_0 n}{\kappa_0+n}(\mu_0-\bar{y})^2 \\ \mu_n &= \tfrac{\kappa_0\mu_0+n\bar{y}}{\kappa_0+n} \\ \kappa_n &= \kappa_0+n\end{aligned}$$

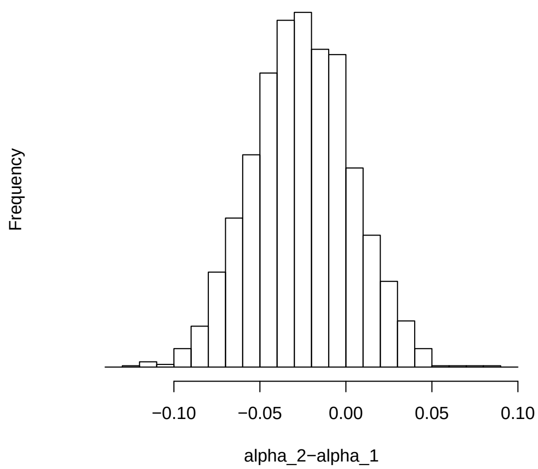
2. Assume independent uniform prior distribution on the multinomial parameters. Then the posterior distributions are independent Dirichlet:

$$\begin{gathered}(\pi_1,\pi_2,\pi_3)|y \sim Dirichlet(295,308,39) \\ (\pi_1^*,\pi_2^*,\pi_3^*)|y \sim Dirichlet(289,333,20),\end{gathered}$$

and $\alpha_1 = \frac{\pi_1}{\pi_1+\pi_2}$, $\alpha_2 = \frac{\pi_1^*}{\pi_1^*+\pi_2^*}$. From the properties of the Dirichlet distribution (see Exercise 3.1),

$$\alpha_1|y \sim Beta(295, 308)$$
$$\alpha_2|y \sim Beta(289, 333).$$

The histogram of 1000 draws from the posterior density of $\alpha_2 - \alpha_1$ is below. Based on this histogram, the posterior probability that there was a shift toward Bush is 19%.




```
alpha.1_rbeta(2000,295,308)
alpha.2_rbeta(2000,289,333)
dif_alpha.2-alpha.1
hist(dif,xlab="alpha_2-alpha_1",yaxt="n",
breaks=seq(-.14,.1,.01),cex=2)
> length(dif[dif>0])/2000
0.1845
```