Introduction to Matrix Calculus

A *matrix* is any rectangular array of real numbers. We denote an arbitrary array of p rows and n columns by,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pn} \end{bmatrix}_{(p \times n)}$$

The *transpose* operation $\mathbf{A}'$ of a matrix changes the columns into rows so that the first column of $\mathbf{A}$ becomes the first row of $\mathbf{A}'$, the second column becomes second row, and so fourth.

Example 1:

If

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 5 & 4 \end{bmatrix}_{(2 \times 3)}$$

then

$$\mathbf{A}' = \begin{bmatrix} 3 & 1 \\ -1 & 5 \\ 2 & 4 \end{bmatrix}_{(3 \times 2)}$$

A matrix may also be multiplied by a constant c. The product $c\mathbf{A}$ is the matrix that results from multiplying each element of $\mathbf{A}$ by c. Thus

$$c\mathbf{A} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{p1} & ca_{p2} & \cdots & ca_{pn} \end{bmatrix}_{(p \times n)}$$

Two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same dimensions can be added. The sum $\mathbf{A} + \mathbf{B}$ has (i, j) entry $a_{ij} + b_{ij}$.

Example 2:

If

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & 1 \\ 1 & -1 & 1 \end{bmatrix}_{(2 \times 3)} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & -2 & -3 \\ 2 & 5 & 1 \end{bmatrix}_{(2 \times 3)},$$

then

$$2\mathbf{A} = \begin{bmatrix} 0 & 6 & 2 \\ 2 & -2 & 2 \end{bmatrix}_{(2\times3)}$$

and

$$\begin{aligned} \mathbf{A}+\mathbf{B} &= \begin{bmatrix} 0+1 & 3-2 & 1-3 \\ 1+2 & -1+5 & 1+1 \end{bmatrix}_{(2\times3)} \\ &= \begin{bmatrix} 1 & 1 & -2 \\ 3 & 4 & 2 \end{bmatrix}_{(2\times3)} \end{aligned}$$

It is possible to define matrix multiplication if the dimensions of the matrices confirm in the following manner. When $\mathbf{A}$ is $(p \times k)$ and $\mathbf{B}$ is $(k \times n)$, so that the number of elements in a row of $\mathbf{A}$ is the same as the number of elements in the columns of $\mathbf{B}$, we can form the matrix product $\mathbf{AB}$. An element of the new matrix $\mathbf{AB}$ is formed by taking the inner product of each row of $\mathbf{A}$ with each column of $\mathbf{B}$. The *matrix product* $\mathbf{AB}$ is

$\mathbf{A_{(p\times k)}B_{(k\times n)}}$ = the $(p \times n)$ matrix whose entry in the i^{th} row of $\mathbf{A}$ and the j^{th} column of $\mathbf{B}$

or

(i,j) entry of $\mathbf{AB} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj} = \sum_{l=1}^{k} a_{il}b_{lj}$

Example 3:
If

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 5 & 4 \end{bmatrix}_{(2\times3)} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -2 \\ 7 \\ 9 \end{bmatrix}_{(3\times1)} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}_{(2\times2)}$$

then and

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 3 & -1 & 2 \\ 1 & 5 & 4 \end{bmatrix}_{(2\times3)} \begin{bmatrix} -2 \\ 7 \\ 9 \end{bmatrix}_{(3\times1)} \\ &= \begin{bmatrix} 3(-2)+(-1)7+2(9) \\ 1(-2)+5(7)+4(9) \end{bmatrix}_{(2\times1)} \\ &= \begin{bmatrix} 5 \\ 69 \end{bmatrix}_{(2\times1)} \end{aligned}$$

Similarly,

$$\mathbf{CA} = \begin{bmatrix} 6 & -2 & 4 \\ 2 & -6 & -2 \end{bmatrix}_{(2\times 3)}$$

Square matrices are of special importance in development of statistical methods. A square matrix is said to be symmetric if $\mathbf{A} = \mathbf{A}'$ or $a_{ij} = a_{ji}$ for all i and j.

Example 4:
The matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 5 \\ 5 & -2 \end{bmatrix}_{(2\times 2)}$$

is symmetric; the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 5 & -2 \end{bmatrix}_{(2\times 2)}$$

is not symmetric

When two square matrices $\mathbf{A}$ and $\mathbf{B}$ are of same dimensions, both products $\mathbf{AB}$ and $\mathbf{BA}$ are defined, although they need not be equal. if we let bf I denote the square matrix with ones on the diagonal and zeros elsewhere it follows from the definition of matrix multiplication that the (i, j) enrty of $\mathbf{AI}$ is $a_{i1} \times 0 + \cdots + a_{i,j-1} \times 0 + a_{ij} \times 1 + a_{i,j+1} \times 0 + \cdots + a_{ik} \times 0 = a_{ij}$, so $\mathbf{AI} = \mathbf{A}$. Similarly, $\mathbf{IA}$ $= \mathbf{A}$ so for any $\mathbf{A}$,

$$\mathbf{I}_{k\times k}\mathbf{A}_{k\times k} = \mathbf{A}_{k\times k}$$

The matrix $\mathbf{I}$ acts like 1 in ordinary multiplication, so it is called identity matrix.

The fundamental scalar relation about the existence of an inverse number a^{-1} such that $a^{-1}a = aa^{-1} = 1$, if $a \neq 0$, has the following matrix algebra extension. If there exists a matrix $\mathbf{B}$ such that

$$\mathbf{B}_{(k\times k)}\mathbf{A}_{(k\times k)} = \mathbf{A}_{(k\times k)}\mathbf{B}_{(k\times k)} = \mathbf{I}_{k\times k}$$

then $\mathbf{B}$ is called the inverse of $\mathbf{A}$ and is denoted by $\mathbf{A}^{-1}$.

Singularity:
A square matrix that not have a matrix inverse is called **singular matrix**. A matrix is singular *iff* its **determinant** is 0. The determinant of a matrix A is denoted as |A|.

Some special matrices:

1) Identity Matrix:

2) Block Matrix: A block matrix is a matrix that is defined using smaller matrices, called blocks. For example, $E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$$

$$C = \begin{bmatrix} 11 & 12 & 13 \\ 14 & 15 & 16 \\ 17 & 18 & 19 \end{bmatrix}$$

$$D = \begin{bmatrix} 20 & 21 \\ 22 & 23 \\ 24 & 25 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 2 & 5 & 6 & 7 \\ 3 & 4 & 8 & 9 & 10 \\ 11 & 12 & 13 & 20 & 21 \\ 14 & 15 & 16 & 22 & 23 \\ 17 & 18 & 19 & 24 & 25 \end{bmatrix}$$

3) Diagonal Matrix: A square matrix of the form

$a_{ij} = c_i \delta_{ij}, where \;\; \delta_{ij} = 1 \;\; if \;\; i = j, \;\; \delta_{ij} = 0 \;\; if \;\; i \neq j.$

A diagonal matrix has the form

$$A = \begin{bmatrix} c1 & 0 & \ldots & 0 \\ 0 & c2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & cn \end{bmatrix}$$

Question:

1) Is identity matrix a diagonal matrix?
2) What is a block-diagonal matrix (PP56, Diggle, Liang, and Zegger, 1994)?

Using matrices to represent simultaneous equations

Example 1:

5x + 3y + z = 1

2x + 3y + 5z = 2

x + 9y + 6z = 3

Rewrite the three equations as a single matrix equation:

AX=V

where $A = \begin{bmatrix} 5 & 3 & 1 \\ 2 & 3 & 5 \\ 1 & 9 & 6 \end{bmatrix}$ $U = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $V = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Example 2:

Linear Regression Model:

$Y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_p x_{ip} + \varepsilon_i$, *i = 1, 2, …, n.*

In Matrix form,

Y=Xβ+e.

Web Resource for Matrix:

http://mathworld.wolfram.com/

It has all the definition related to matrix and matrix operation.

Exercise:

ρ = 0.5

I is 5x5 identity matrix

J is 5x5 matrix with all of its elements of 1.

$$X = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Calculate

1) $V_0 = (1-\rho)I+\rho J$
2) X'X
3) X' V_0 X