

## Introduction to Matrix Calculus

A *matrix* is any rectangular array of real numbers. We denote an arbitrary array of  $p$  rows and  $n$  columns by,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pn} \end{bmatrix}_{(p \times n)}$$

The *transpose* operation  $\mathbf{A}'$  of a matrix changes the columns into rows so that the first column of  $\mathbf{A}$  becomes the first row of  $\mathbf{A}'$ , the second column becomes second row, and so fourth.

### Example 1:

If

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 5 & 4 \end{bmatrix}_{(2 \times 3)}$$

then

$$\mathbf{A}' = \begin{bmatrix} 3 & 1 \\ -1 & 5 \\ 2 & 4 \end{bmatrix}_{(3 \times 2)}$$

A matrix may also be multiplied by a constant  $c$ . The product  $c\mathbf{A}$  is the matrix that results from multiplying each element of  $\mathbf{A}$  by  $c$ . Thus

$$c\mathbf{A} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{p1} & ca_{p2} & \cdots & ca_{pn} \end{bmatrix}_{(p \times n)}$$

Two matrices  $\mathbf{A}$  and  $\mathbf{B}$  of the same dimensions can be added. The sum  $\mathbf{A} + \mathbf{B}$  has  $(i, j)$  entry  $a_{ij} + b_{ij}$ .

### Example 2:

If

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & 1 \\ 1 & -1 & 1 \end{bmatrix}_{(2 \times 3)} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & -2 & -3 \\ 2 & 5 & 1 \end{bmatrix}_{(2 \times 3)},$$

then

$$2\mathbf{A} = \begin{bmatrix} 0 & 6 & 2 \\ 2 & -2 & 2 \end{bmatrix}_{(2 \times 3)}$$

and

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \begin{bmatrix} 0+1 & 3-2 & 1-3 \\ 1+2 & -1+5 & 1+1 \end{bmatrix}_{(2 \times 3)} \\ &= \begin{bmatrix} 1 & 1 & -2 \\ 3 & 4 & 2 \end{bmatrix}_{(2 \times 3)} \end{aligned}$$

It is possible to define matrix multiplication if the dimensions of the matrices confirm in the following manner. When  $\mathbf{A}$  is  $(p \times k)$  and  $\mathbf{B}$  is  $(k \times n)$ , so that the number of elements in a row of  $\mathbf{A}$  is the same as the number of elements in the columns of  $\mathbf{B}$ , we can form the matrix product  $\mathbf{AB}$ . An element of the new matrix  $\mathbf{AB}$  is formed by taking the inner product of each row of  $\mathbf{A}$  with each column of  $\mathbf{B}$ . The *matrix product*  $\mathbf{AB}$  is

$\mathbf{A}_{(p \times k)}\mathbf{B}_{(k \times n)}$  = the  $(p \times n)$  matrix whose entry in the  $i^{th}$  row of  $\mathbf{A}$  and the  $j^{th}$  column of  $\mathbf{B}$

or

$(i, j)$  entry of  $\mathbf{AB} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{ik}b_{kj} = \sum_{l=1}^k a_{il}b_{lj}$

**Example 3:**

If

$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 5 & 4 \end{bmatrix}_{(2 \times 3)} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -2 \\ 7 \\ 9 \end{bmatrix}_{(3 \times 1)} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}_{(2 \times 2)}$$

then and

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 3 & -1 & 2 \\ 1 & 5 & 4 \end{bmatrix}_{(2 \times 3)} \begin{bmatrix} -2 \\ 7 \\ 9 \end{bmatrix}_{(3 \times 1)} \\ &= \begin{bmatrix} 3(-2) + (-1)7 + 2(9) \\ 1(-2) + 5(7) + 4(9) \end{bmatrix}_{(2 \times 1)} \\ &= \begin{bmatrix} 5 \\ 69 \end{bmatrix}_{(2 \times 1)} \end{aligned}$$

Similarly,

$$\mathbf{CA} = \begin{bmatrix} 6 & -2 & 4 \\ 2 & -6 & -2 \end{bmatrix}_{(2 \times 3)}$$

Square matrices are of special importance in development of statistical methods. A square matrix is said to be symmetric if  $\mathbf{A} = \mathbf{A}'$  or  $a_{ij} = a_{ji}$  for all  $i$  and  $j$ .

**Example 4:**

The matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 5 \\ 5 & -2 \end{bmatrix}_{(2 \times 2)}$$

is symmetric; the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 5 & -2 \end{bmatrix}_{(2 \times 2)}$$

is not symmetric

When two square matrices  $\mathbf{A}$  and  $\mathbf{B}$  are of same dimensions, both products  $\mathbf{AB}$  and  $\mathbf{BA}$  are defined, although they need not be equal. If we let  $\mathbf{I}$  denote the square matrix with ones on the diagonal and zeros elsewhere it follows from the definition of matrix multiplication that the  $(i, j)$  entry of  $\mathbf{AI}$  is  $a_{i1} \times 0 + \dots + a_{i,j-1} \times 0 + a_{ij} \times 1 + a_{i,j+1} \times 0 + \dots + a_{ik} \times 0 = a_{ij}$ , so  $\mathbf{AI} = \mathbf{A}$ . Similarly,  $\mathbf{IA} = \mathbf{A}$  so for any  $\mathbf{A}$ ,

$$\mathbf{I}_{k \times k} \mathbf{A}_{k \times k} = \mathbf{A}_{k \times k}$$

The matrix  $\mathbf{I}$  acts like 1 in ordinary multiplication, so it is called identity matrix.

The fundamental scalar relation about the existence of an inverse number  $a^{-1}$  such that  $a^{-1}a = aa^{-1} = 1$ , if  $a \neq 0$ , has the following matrix algebra extension. If there exists a matrix  $\mathbf{B}$  such that

$$\mathbf{B}_{(k \times k)} \mathbf{A}_{(k \times k)} = \mathbf{A}_{(k \times k)} \mathbf{B}_{(k \times k)} = \mathbf{I}_{k \times k}$$

then  $\mathbf{B}$  is called the inverse of  $\mathbf{A}$  and is denoted by  $\mathbf{A}^{-1}$ .

**Singularity:**

A square matrix that not have a matrix inverse is called **singular matrix**. A matrix is singular *iff* its **determinant** is 0. The determinant of a matrix A is denoted as  $|A|$ .

**Some special matrices:**

1) Identity Matrix:

2) Block Matrix: A block matrix is a matrix that is defined using smaller

matrices, called blocks. For example,  $E = \begin{matrix} A & B \\ C & D \end{matrix}$

$$A = \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}$$

$$B = \begin{matrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{matrix}$$

$$C = \begin{matrix} 11 & 12 & 13 \\ 14 & 15 & 16 \\ 17 & 18 & 19 \end{matrix}$$

$$D = \begin{matrix} 20 & 21 \\ 22 & 23 \\ 24 & 25 \end{matrix}$$

$$E = \begin{matrix} 1 & 2 & 5 & 6 & 7 \\ 3 & 4 & 8 & 9 & 10 \\ 11 & 12 & 13 & 20 & 21 \\ 14 & 15 & 16 & 22 & 23 \\ 17 & 18 & 19 & 24 & 25 \end{matrix}$$

3) Diagonal Matrix: A square matrix of the form

$$a_{ij} = c_i \delta_{ij}, \text{ where } \delta_{ij} = 1 \text{ if } i = j, \delta_{ij} = 0 \text{ if } i \neq j.$$

A diagonal matrix has the form

$$A = \begin{pmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & c_n \end{pmatrix}$$

Question:

- 1) Is identity matrix a diagonal matrix?
- 2) What is a block-diagonal matrix (PP56, Diggle, Liang, and Zegger, 1994)?

### Using matrices to represent simultaneous equations

Example 1:

$$5x + 3y + z = 1$$

$$2x + 3y + 5z = 2$$

$$x + 9y + 6z = 3$$

Rewrite the three equations as a single matrix equation:

$$AX = V$$

$$\begin{pmatrix} 5 & 3 & 1 \\ 2 & 3 & 5 \\ 1 & 9 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

where  $A = \begin{pmatrix} 5 & 3 & 1 \\ 2 & 3 & 5 \\ 1 & 9 & 6 \end{pmatrix}$   $U = y$   $V = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ .

Example 2:

Linear Regression Model:

$$Y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \dots + \beta_p x_{ip} + \varepsilon_i, i = 1, 2, \dots, n.$$

In Matrix form,

$$Y = X + e.$$

Web Resource for Matrix:

<http://mathworld.wolfram.com/>

It has all the definition related to matrix and matrix operation.

**Exercise:**

$$= 0.5$$

I is 5x5 identity matrix

J is 5x5 matrix with all of its elements of 1.

$$1 \quad -2$$

$$1 \quad -1$$

$$X = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$1 \quad 1$$

$$1 \quad 2$$

Calculate

1)  $V_0 = (I - \frac{1}{5}J)$

2)  $X'X$

3)  $X' V_0 X$