Lecture 14

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Table of contents

1. Table of contents
2. Outline
3. Logs
4. The geometric mean
5. GM and the CLT
6. Comparisons
7. The log-normal distribution

Table of contents

Outline

Logs

The geometric mean

GM and the CLT

Comparisons

The log-normal distribution

Outline

1. Review about logs
2. Introduce the geometric mean
3. Interpretations of the geometric mean
4. Confidence intervals for the geometric mean
5. Log-normal distribution
6. Log-normal based intervals

Logs

- Recall that $\log_B(x)$ is the number y so that $B^y = x$
- Note that you can not take the log of a negative number; $\log_B(1)$ is always 0 and $\log_B(0)$ is $-\infty$
- When the base is $B = e$ we write $\log_e$ as just log or ln
- Other useful bases are 10 (orders of magnitude) or 2
- Recall that $\log(ab) = \log(a) + \log(b)$, $\log(a^b) = b\log(a)$, $\log(a/b) = \log(a) - \log(b)$ (log turns multiplication into addition, division into subtraction, powers into multiplication)

Table of contents

Outline

Logs

The geometric mean

GM and the CLT

Comparisons

The log-normal distribution

Some reasons for “logging” data

- To correct for right skewness
- When considering ratios
- In settings where errors are feasibly multiplicative, such as when dealing with concentrations or rates
- To consider orders of magnitude (using log base 10); for example when considering astronomical distances
- Counts are often logged (though note the problem with zero counts)

Table of contents

Outline

Logs

The geometric mean

GM and the CLT

Comparisons

The log-normal distribution

The geometric mean

- The (sample) **geometric mean** of a data set $X_1, \ldots, X_n$ is

$$\left(\prod_{i=1}^{n} X_i\right)^{1/n}$$

- Note that (provided that the X_i are positive) the log of the geometric mean is

$$\frac{1}{n}\sum_{i=1}^{n} \log(X_i)$$

- As the log of the geometric mean is an average, the LLN and clt apply (under what assumptions?)
- The geometric mean is always less than or equal to the sample (arithmetic) mean

The geometric mean

- The geometric mean is often used when the X_i are all multiplicative
- Suppose that in a population of interest, the prevalence of a disease rose 2% one year, then fell 1% the next, then rose 2%, then rose 1%; since these factors act multiplicatively it makes sense to consider the geometric mean

$$(1.02 \times .99 \times 1.02 \times 1.01)^{1/4} = 1.01$$

  for a 1% geometric mean increase in disease prevalence

- Notice that multiplying the initial prevalence by 1.01^4 is the same as multiplying by the original four numbers in sequence
- Hence 1.01 is constant factor by which you would need to multiply the initial prevalence each year to achieve the same overall increase in prevalence over a four year period
- The arithmetic mean, in contrast, is the constant factor by which your would need to *add* each year to achieve the same *total* increase $(1.02 + .99 + 1.02 + 1.01)$
- In this case the product and hence the geometric mean make more sense than the arithmetic mean

Nifty fact

- The *question corner* (google) at the University of Toronto's web site (where I got much of this) has a fun interpretation of the geometric mean
- If a and b are the lengths of the sides of a rectangle then
  - The arithmetic mean $(a+b)/2$ is the length of the sides of the square that has the same perimeter
  - The geometric mean $(ab)^{1/2}$ is the length of the sides of the square that has the same area
- So if you're interested in perimeters (adding) use the arithmetic mean; if you're interested in areas (multiplying) use the geometric mean

Table of contents

Outline

Logs

The geometric mean

GM and the CLT

Comparisons

The log-normal distribution

Asymptotics

- Note, by the LLN the log of the geometric mean converges to $\mu = E[\log(X)]$
- Therefore the geometric mean converges to $\exp\{E[\log(X)]\} = e^{\mu}$, which is *not* the population mean on the natural scale; we call this the population geometric mean (but no one else seems to)
- To reiterate

$$\exp\{E[\log(x)]\} \neq E[\exp\{\log(X)\}] = E[X]$$

- Note if the distribution of $\log(X)$ is symmetric then

$$.5 = P(\log X \leq \mu) = P(X \leq e^{\mu})$$

- Therefore, for log-symmetric distributions the geometric mean is estimating the median

Using the CLT

- If you use the CLT to create a confidence interval for the log measurements, your interval is estimating μ, the expected value of the log measurements
- If you exponentiate the endpoints of the interval, you are estimating e^{μ}, the population geometric mean
- Recall, e^{μ} is the population median when the distribution of the logged data is symmetric
- This is especially useful for paired data when their ratio, rather than their difference, is of interest

Table of contents

Outline

Logs

The geometric mean

GM and the CLT

Comparisons

The log-normal distribution

Comparisons

- Consider when you have two independent groups, logging the individual data points and creating a confidence interval for the difference in the log means
- Prove to yourself that exponentiating the endpoints of this interval is then an interval for the *ratio* of the population geometric means, $\frac{e^{\mu_1}}{e^{\mu_2}}$

Table of contents

Outline

Logs

The geometric mean

GM and the CLT

Comparisons

The log-normal distribution

The log-normal distribution

- A random variable is **log-normally** distributed *if its log is a normally distributed random variable*
- "I am log-normal" means "take logs of me and then I'll then be normal"
- Note log-normal random variables are not logs of normal random variables!!!!!! (You can't even take the log of a normal random variable)
- Formally, X is lognormal(μ, σ^2) if $\log(X) \sim \mathrm{N}(\mu, \sigma^2)$
- If $Y \sim \mathrm{N}(\mu, \sigma^2)$ then $X = e^Y$ is log-normal

Table of contents
Outline
Logs
The geometric mean
GM and the CLT
Comparisons
The log-normal distribution

The log-normal distribution

- The log-normal density is

$$\frac{1}{\sqrt{2\pi}} \times \frac{\exp[-\{\log(x) - \mu\}^2/(2\sigma^2)]}{x} \quad \text{for} \quad 0 \leq x \leq \infty$$

- Its mean is $e^{\mu + (\sigma^2/2)}$ and variance is $e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$
- Its median is e^{μ}

Table of contents
Outline
Logs
The geometric mean
GM and the CLT
Comparisons
The log-normal distribution

The log-normal distribution

- Notice that if we assume that $X_1, \ldots, X_n$ are log-normal(μ, σ^2) then $Y_1 = \log X_1, \ldots, Y_n = \log X_n$ are normally distributed with mean μ and variance σ^2
- Creating a Gosset's t confidence interval on using the Y_i is a confidence interval for μ the log of the median of the X_i
- Exponentiate the endpoints of the interval to obtain a confidence interval for e^μ, the median on the original scale
- Assuming log-normality, exponentiating t confidence intervals for the difference in two log means again estimates ratios of geometric means

Example: interpret these results

Gray matter volumes for 342 older subjects (over 60) and 287 younger subjects were compared.

- The mean log gray matter volumes was 6.35 $\log(\mathrm{cm}^3)$ (older) and 6.40 $\log(\mathrm{cm}^3)$ (younger). Exponentiating these numbers leads to 570.90 cm^3 and 599.40 cm^3
- The SDs were 0.11 $\log(\mathrm{cm}^3)$ and 0.11 $\log(\mathrm{cm}^3)$
- CIs
  - Younger: log scale - $[6.38, 6.41]$, exponentiated - $[592.03, 606.86]$
  - Older: log scale - $[6.34, 6.36]$, exponentiated - $[564.36, 577.50]$
- Two sample mean comparison
  - Log scale - $[0.03, 0.07]$
  - Exponentiated - $[1.03, 1.07]$