Lecture 24

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1. Odds ratios for retrospective studies
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Case-control methods

	Lung cancer		
Smoker	Cases	Controls	Total
Yes	688	650	1338
No	21	59	80
	709	709	1418

- Case status obtained from records
- Cannot estimate $P(\text{Case} \mid \text{Smoker})$
- Can estimate $P(\text{Smoker} \mid \text{Case})$

Continued

- Can estimate odds ratio b/c

$$\frac{Odds(\text{case} \mid \text{smoker})}{Odds(\text{case} \mid \text{smoker}^c)} = \frac{Odds(\text{smoker} \mid \text{case})}{Odds(\text{smoker} \mid \text{case}^c)}$$

Proof

C - case, S - smoker

$$
\begin{aligned}
& \frac{Odds(\text{case} \mid \text{smoker})}{Odds(\text{case} \mid \text{smoker}^c)} \\
= & \frac{P(C \mid S)/P(\bar{C} \mid S)}{P(C \mid \bar{S})/P(\bar{C} \mid \bar{S})} \\
= & \frac{P(C, S)/P(\bar{C}, S)}{P(C, \bar{S})/P(\bar{C}, \bar{S})} \\
= & \frac{P(C, S)P(\bar{C}, \bar{S})}{P(C, \bar{S})P(\bar{C}, S)}
\end{aligned}
$$

Exchange C and S and the result is obtained

Notes

- Sample OR is $\frac{n_{11}n_{22}}{n_{12}n_{21}}$
- Sample OR is unchanged if a row or column is multiplied by a constant
- Invariant to transposing
- Is related to RR

Notes continued

$$\begin{aligned}
OR &= \frac{P(S \mid C)/P(\bar{S} \mid C)}{P(S \mid \bar{C})/P(\bar{S} \mid \bar{C})} \\
&= \frac{P(C \mid S)/P(\bar{C} \mid S)}{P(C \mid \bar{S})/P(\bar{C} \mid \bar{S})} \\
&= \frac{P(C \mid S)}{P(C \mid \bar{S})} \frac{P(\bar{C} \mid \bar{S})}{P(\bar{C} \mid S)} \\
&= RR \times \frac{1 - P(C \mid \bar{S})}{1 - P(C \mid S)}
\end{aligned}$$

- OR approximate RR if $P(C \mid \bar{S})$ and $P(C \mid S)$ are small (or if they are nearly equal)

Rare disease assumption

	Disease		
Exposure	Yes	No	Total
Yes	9	1	10
No	1	999	1000
	10	1000	1010

- Cross-sectional data
- $P(\hat{D}) = 10/1010 \approx .01$
- $\hat{OR} = (9 \times 999)/(1 \times 1) = 8991$
- $\hat{RR} = (9/10)/(1/1000) = 900$
- D is rare in the sample
- D is not rare among the exposed

Notes

- $OR = 1$ implies no association
- $OR > 1$ positive association
- $OR < 1$ negative association
- For retrospective CC studies, OR can be interpreted prospectively
- For diseases that are rare among the cases and controls, the OR approximates the RR
- Delta method SE for log OR is

$$\sqrt{\frac{1}{n_{11}} + \frac{1}{n_{12}} + \frac{1}{n_{21}} + \frac{1}{n_{22}}}$$

Example

	Lung cancer		
Smoker	Cases	Controls	Total
Yes	688	650	1338
No	21	59	80
	709	709	1418

[1]

- $\hat{OR} = \frac{688 \times 59}{21 \times 650} = 3.0$
- $\hat{SE}_{\log \hat{OR}} = \sqrt{\frac{1}{688} + \frac{1}{650} + \frac{1}{21} + \frac{1}{59}} = .26$
- $CI = \log(3.0) \pm 1.96 \times .26 = [.59, 1.61]$
- The estimated odds of lung cancer for smokers are 3 times that of the odds for non-smokers with an interval of $[\exp(.59), \exp(1.61)] = [1.80, 5.00]$

[1] Data from Agresti, Categorical Data Analysis, second edition

Exact inference for the OR

	Lung cancer		
Smoker	Cases	Controls	Total
Yes	688	650	1338
No	21	59	80
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- X the number of smokers for the cases
- Y the number of smokers for the controls
- Calculate an exact CI for the odds ratio
- Have to eliminate a nuisance parameter

Notation

- $\mathrm{logit}(p) = \log\{p/(1-p)\}$ is the **log-odds**
- Differences in $\mathrm{logit}s$ are log-odds *ratios*
- $\mathrm{logit}\{P(\text{Smoker} \mid \text{Case})\} = \delta$
    - $P(\text{Smoker} \mid \text{Case}) = e^{\delta}/(1+e^{\delta})$
- $\mathrm{logit}\{P(\text{Smoker} \mid \text{Control})\} = \delta + \theta$
    - $P(\text{Smoker} \mid \text{Control}) = e^{\delta+\theta}/(1+e^{\delta+\theta})$
- θ is the log-odds ratio
- δ is the nuisance parameter

Notation

- X is binomial with n_1 trials and success probability $e^{\delta}/(1+e^{\delta})$
- Y is binomial with n_2 trials and success probability $e^{\delta+\theta}/(1+e^{\delta+\theta})$

$$
\begin{aligned}
P(X=x) &= \binom{n_1}{x}\left\{\frac{e^{\delta}}{1+e^{\delta}}\right\}^{x}\left\{\frac{1}{1+e^{\delta}}\right\}^{n_1-x} \\
&= \binom{n_1}{x} e^{x\delta}\left\{\frac{1}{1+e^{\delta}}\right\}^{n_1}
\end{aligned}
$$

$$P(X = x) = \binom{n_1}{x} e^{x\delta} \left\{ \frac{1}{1 + e^{\delta}} \right\}^{n_1}$$

$$P(Y = z - x) = \binom{n_2}{z - x} e^{(z-x)\delta + (z-x)\theta} \left\{ \frac{1}{1 + e^{\delta + \theta}} \right\}^{n_2}$$

$$P(X + Y = z) = \sum_u P(X = u) P(Y = z - u)$$

$$P(X = x \mid X + Y = z) = \frac{P(X = x)P(Y = z - x)}{\sum_u P(X = u)P(Y = z - u)}$$

Non-central hypergeometric distribution

$$P(X = x \mid X + Y = z; \theta) = \frac{\binom{n_1}{x}\binom{n_2}{z-x} e^{x\theta}}{\sum_u \binom{n_1}{u}\binom{n_2}{z-u} e^{u\theta}}$$

- θ is the log odds ratio
- This distribution is used to calculate exact hypothesis tests for $H_0 : \theta = \theta_0$
- Inverting exact tests yields exact confidence intervals for the odds ratio
- Simplifies to the hypergeometric distribution for $\theta = 0$