10 Estimable Functions and the Gauss-Markov Theorem

A quick review:

10.1 Definition: We defined the Best Linear Unbiased Estimate (BLUE) of a parameter θ based on data $\mathbf{Y}$

(a) is a linear combination of $\mathbf{Y}$ (i. e. equals $\mathbf{b}'\mathbf{Y}$),

(b) is unbiased (i. e. $E[\mathbf{b}'\mathbf{Y}] = \theta$),

(c) has the smallest variance among all such unbiased linear estimators.

10.2 Theorem: For any linear combination $\mathbf{c}'E[\mathbf{Y}]$, $\mathbf{c}'\hat{\mathbf{Y}}$ is the BLUE of $\mathbf{c}'E[\mathbf{Y}]$, where $\hat{\mathbf{Y}}$ is the least-squares orthogonal projection of $\mathbf{Y}$.

10.3 Theorem: If rank$(\mathbf{X}_{n\times p}) = p$, then $\mathbf{a}'\hat{\boldsymbol{\beta}}$ is the BLUE of $\mathbf{a}'\boldsymbol{\beta}$ for any $\mathbf{a}$.

10.4 Note: In the less than full rank case, there is not a unique way to estimate $\boldsymbol{\beta}$. However, we will see that certain linear combinations of the components of $\boldsymbol{\beta}$ can be unbiasedly estimated.

10.5 Definition: A linear combination $\mathbf{a}'\boldsymbol{\beta}$ is estimable if it has a linear unbiased estimate, i.e., $\mathbf{b}'\mathbf{Y}$ for some $\mathbf{b}$ such that $E[\mathbf{b}'\mathbf{Y}] = \mathbf{a}'\boldsymbol{\beta}$ for all $\boldsymbol{\beta}$.

10.6 Theorem: $\mathbf{a}'\boldsymbol{\beta}$ is estimable if and only if $\mathbf{a} \in \mathcal{R}(\mathbf{X}')$.

10.7 Theorem: If $\mathbf{a}'\boldsymbol{\beta}$ is estimable, there is a unique $\mathbf{b}_* \in \mathcal{R}(\mathbf{X})$ such that $\mathbf{a} = \mathbf{X}'\mathbf{b}_*$.

10.8 Note: In the full rank case any $\mathbf{a}'\boldsymbol{\beta}$ is estimable. In particular, $\mathbf{a}'\hat{\boldsymbol{\beta}} = \mathbf{a}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \equiv \mathbf{b}'\mathbf{Y}$ is a linear unbiased estimate of $\mathbf{a}'\boldsymbol{\beta}$. In this case we also know that $\mathbf{a}'\hat{\boldsymbol{\beta}}$ is the BLUE.

10.9 Theorem: (Gauss-Markov): If $\mathbf{a}'\boldsymbol{\beta}$ is estimable, then

(a) $\mathbf{a}'\hat{\boldsymbol{\beta}}$ is unique (i.e. the same for all solutions $\hat{\boldsymbol{\beta}}$ to the normal equations),

(b) $\mathbf{a}'\hat{\boldsymbol{\beta}}$ is the BLUE of $\mathbf{a}'\boldsymbol{\beta}$.

10.10 Theorem: If $\mathbf{a}'\boldsymbol{\beta}$ is estimable then $\mathbf{a}'(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{X} = \mathbf{a}'$ for any generalized inverse $(\mathbf{X}'\mathbf{X})^{-}$.

10.11 Theorem: If $\mathbf{a}'\boldsymbol{\beta}$ is estimable, then $\text{var}(\mathbf{a}'\hat{\boldsymbol{\beta}}) = \sigma^2\mathbf{a}'(\mathbf{X}'\mathbf{X})^{-}\mathbf{a}$.