2.9 Vector Valued Random Variables

Definition: If $\mathbf{Y}$ is an $n \times 1$ vector of random variables we define the expected value of $\mathbf{Y}$ as

$$E(\mathbf{Y}) = \begin{bmatrix} E(Y_1) \\ E(Y_2) \\ \vdots \\ E(Y_n) \end{bmatrix}$$

i.e. $E(\mathbf{Y})$ is the $n \times 1$ vector with ith coordinate equal to the expected value of the ith coordinate of $\mathbf{Y}$.

Definition: If $\mathbf{Y}$ is a $n \times 1$ vector of random variables we define the variance covariance matrix of $\mathbf{Y}$ as

$$\text{var}\,(\mathbf{Y}) = E\left\{[\mathbf{Y} - E(\mathbf{Y})]\,[\mathbf{Y} - E(\mathbf{Y})]^T\right\}$$

Note that

$$\begin{aligned} \text{var}\,(\mathbf{Y}) &= E\left\{[\mathbf{Y} - E(\mathbf{Y})]\,[\mathbf{Y} - E(\mathbf{Y})]^T\right\} \\ &= E\left\{\mathbf{Y}\mathbf{Y}^T\right\} - [E(\mathbf{Y})]\,[E(\mathbf{Y})]^T \\ &= \{(\text{cov}\,(Y_i, Y_j))\} \end{aligned}$$

i.e. the i, j element of var $(\mathbf{Y})$ is equal to cov (Y_i, Y_j).

Similarly, if $\mathbf{Y}$ is $n \times 1$ and $\mathbf{X}$ is $p \times 1$ then the covariance matrix of $\mathbf{Y}$ and $\mathbf{X}$ is the $n \times p$ matrix defined as

$$\text{cov}\,(\mathbf{Y}, \mathbf{X}) = E\left\{[\mathbf{Y} - E(\mathbf{Y})]\,[\mathbf{X} - E(\mathbf{X})]^T\right\}$$

Note that

$$\begin{aligned} \text{cov}\,(\mathbf{Y}, \mathbf{X}) &= E\left\{[\mathbf{Y} - E(\mathbf{Y})]\,[\mathbf{X} - E(\mathbf{X})]^T\right\} \\ &= E\left\{\mathbf{Y}\mathbf{X}^T\right\} - [E(\mathbf{Y})]\,[E(\mathbf{X})]^T \\ &= \{(\text{cov}\,(Y_i, X_j))\} \end{aligned}$$

If $\mathbf{X}$ and $\mathbf{Y}$ are random variables of appropriate dimensions then

$$\begin{aligned} E(\mathbf{a} + \mathbf{B}\mathbf{Y}) &= \mathbf{a} + \mathbf{B}E(\mathbf{Y}) \\ \text{var}(\mathbf{a} + \mathbf{B}\mathbf{Y}) &= \mathbf{B}\text{var}(\mathbf{Y})\mathbf{B}^T \\ \text{cov}(\mathbf{a} + \mathbf{B}\mathbf{X}, \mathbf{c} + \mathbf{D}\mathbf{Y}) &= \mathbf{B}\text{cov}(\mathbf{X}, \mathbf{Y})\mathbf{D}^T \end{aligned}$$