2.11 Other Distributional Results and Quadratic Forms

Definition: A random variable $U$ is said to have a non-central chi square distribution with $\nu$ degrees of freedom and non-centrality parameter $\delta$ if it can be expressed as $U = \sum_{i=1}^{\nu} Z_i^2$ where $(Z_1, \ldots, Z_\nu)$ are independent Gaussian random variables with unit variance and $\delta = \left[ \sum_{i=1}^{\nu} E(Z_i)^2 \right]^{\frac{1}{2}}$. We write

$$U \sim \chi^2(\nu, \delta)$$

Note that an ordinary or central chi square distribution is the special case of the non-central distribution with $\delta = 0$. Note also that there are several ways that the non-centrality parameter is defined in the literature. Here, it is defined as the length of the mean vector of the $Z_i$'s. Other authors define a non-centrality parameter, $\lambda$, as the square of the length (i.e. $\lambda = \delta^2$) or as one half times the square of the length (i.e. $\lambda = \frac{\delta^2}{2}$).
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One can always choose \( E(Z_1) = \delta \) and \( E(Z_i) = 0 \) for \( 2 \leq i \leq \nu \) so that \( U \) may be represented as the sum of a chi square variable with one degree of freedom and non-centrality parameter \( \delta \) and an independent central chi square with \( \nu - 1 \) degrees of freedom. From this representation, it is easy to derive the first two moments of \( U \):

\[
E(U) = \nu + \delta^2
\]

\[
\text{var}(U) = 2\nu + 4\delta^2
\]

Also if \( U_1 \sim \chi^2(\nu_1, \delta_1) \), and \( U_2 \sim \chi^2(\nu_2, \delta_2) \), with \( U_1, U_2 \) independent, we have \( U_1 + U_2 \sim \chi^2(\nu_1 + \nu_2, (\delta_1^2 + \delta_2^2)\frac{1}{2}) \)

**Definition:** A random variable \( V \) is said to have a non central \( F \)-distribution with \( \nu_1 \) and \( \nu_2 \) degrees of freedom and non-centrality parameter \( \delta \) if it may be written as

\[
V = \frac{U_1}{U_2}
\]

where \( U_1 \) and \( U_2 \) are independent random variables with \( U_1 \sim \chi^2(\nu_1, \delta^2) \) and \( U_2 \sim \chi^2(\nu_2, 0) \). We write \( V \sim F(\nu_1, \nu_2, \delta) \).
Let $Z_1, Z_2, \ldots, Z_p$ be independent $N(0, 1)$ then

$$f_Z(z) = (2\pi)^{-\frac{p}{2}} \exp \left\{ -\frac{1}{2} z^T z \right\}$$

and if $Y = \mu + BZ$ where $BB^T = V$ then $Y$ has a MVN $(\mu, V)$ distribution. The expected value of $Y$ is $\mu$ and the variance covariance matrix of $Y$ is $V$. The density function of $Y$ is given by

$$f_Y(y) = (2\pi)^{-\frac{p}{2}} [\text{det}(V)]^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (y - \mu)^T V^{-1} (y - \mu) \right\}$$

where we assume that $V$ is of rank $p$.

From the definition the m.g.f. of $Z$ is

$$M_Z(t) = \exp \left\{ \frac{1}{2} t^T t \right\}$$

and the m.g.f. of $Y$ is

$$M_Y(t) = \exp \left\{ t^T \mu + \frac{1}{2} t^T V t \right\}$$
Again using the definition we have that

\[ \int (2\pi)^{-\frac{d}{2}} [\det(V)]^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (y - \mu)^T V^{-1} (y - \mu) \right\} \, dy = 1 \]

and hence

\[ \int \exp \left\{ -\frac{1}{2} y^T V^{-1} y + \mu^T V^{-1} y \right\} \, dy = (2\pi)^{\frac{d}{2}} [\det(V)]^{\frac{1}{2}} \exp \left\{ \frac{1}{2} \mu^T V^{-1} \mu \right\} \]

We thus have the fundamental identity

\[ \int \exp \left\{ -\frac{1}{2} y^T \Sigma^{-1} y + \ell^T y \right\} \, dy = (2\pi)^{\frac{d}{2}} [\det(\Sigma)]^{\frac{1}{2}} \exp \left\{ \frac{1}{2} \ell^T \Sigma \ell \right\} \]

which is valid for any positive definite matrix \( \Sigma \) and any \( \ell \).

Now consider the joint m.g.f. of

\[ W = a + LY, \quad Q_1 = Y^T B_1 Y \text{ and } Q_2 = Y^T B_2 Y \]

where \( L \) is \( r \times p \) of rank \( r \) and \( B_1 \) and \( B_2 \) are symmetric non negative definite matrices of ranks \( r_1 \) and \( r_2 \) respectively. By definition the joint m.g.f. is given by

\[ \int \exp \left\{ t^T (Ly + a) + \theta_1 y^T B_1 y + \theta_2 y^T B_2 y \right\} f_Y(y) \, dy \]

which is equal to
\[
(2\pi)^{-\frac{n}{2}}[\det(V)]^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2}(\mu^T V^{-1} \mu + t^T a) \right\} \\
\times \left[ \int \exp \left\{ (t^T L + \mu^T V^{-1})y - \frac{1}{2}y^T [V^{-1} - 2\theta_1 B_1 - 2\theta_2 B_2]y \right\} dy \right]
\]

Using the fundamental identity the joint m.g.f. \( M_{W,Q_1,Q_2}(t, \theta_1, \theta_2) \) is given by

\[
\left[ [\det(V)]^{-\frac{1}{2}}[\det(V^{-1} - 2\theta_1 B_1 - 2\theta_2 B_2)]^{-\frac{1}{2}} \right] \\
\times \left[ \exp \left\{ -\frac{1}{2}\mu^T V^{-1} \mu + t^T a \right\} \right] \\
\times \left[ \exp \left\{ \frac{1}{2}(t^T L + \mu^T V^{-1})[V^{-1} - 2\theta_1 B_1 - 2\theta_2 B_2]^{-1}(L^T t + V^{-1} \mu) \right\} \right]
\]

Now note that

\[
\det(V^{-1} - 2\theta_1 B_1 - 2\theta_2 B_2) = \det(V^{-1}) \det(I - 2\theta_1 B_1 V - 2\theta_2 B_2 V)
\]

\[
[V^{-1} - 2\theta_1 B_1 - 2\theta_2 B_2]^{-1} = V[I - 2\theta_1 B_1 V - 2\theta_2 B_2 V]^{-1}
\]

\[
[I - 2\theta_1 B_1 V - 2\theta_2 B_2 V]^{-1} = \sum_{r=0}^{\infty} 2^r (\theta_1 B_1 V + \theta_2 B_2 V)^r
\]
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The last identity is called the Von-Neuman expansion and is valid if the largest characteristic root of the matrix $\theta_1 B_1 V + \theta_2 B_2$ is less than one. Since $\theta_1$ and $\theta_2$ are arbitrary positive numbers this condition can be assumed.

Using these identities we have that

$$(t^T L + \mu^T V^{-1})[V^{-1} - 2\theta_1 B_1 - 2\theta_2 B_2]^{-1}(L^T t + V^{-1} \mu)$$

is given by

$$(t^T L + \mu^T V^{-1})V(L^T t + V^{-1} \mu) + (t^T L V + \mu^T) [\sum_{r=1}^{\infty} 2^r (\theta_1 B_1 V + \theta_2 B_2 V)^r](L^T t + V^{-1} \mu)$$

The joint m.g.f. of $W$, $Q_1$ and $Q_2$ is thus

$$\left[\det(I - 2\theta_1 B_1 V - 2\theta_2 B_2 V)\right]^{-\frac{1}{2}} \left[\exp \left\{t^T (a + L \mu) + \frac{1}{2} t^T (L V L^T) t\right\}\right] \times \exp \left\{\frac{1}{2}(t^T L V + \mu^T) [\sum_{r=1}^{\infty} 2^r (\theta_1 B_1 V + \theta_2 B_2 V)^r](L^T t + V^{-1} \mu)\right\}$$
2.11.2 Results of Importance in Linear Models

(1) If $\theta_1$ and $\theta_2$ are both 0 the m.g.f. of $W$ is

$$\exp\left\{t^T(a + L\mu) + \frac{1}{2}t^T(LVL)^Tt\right\}$$

which shows that $W$ is MVN $(a + L\mu, LVL^T)$.

(2) If $t = 0$ then the joint m.g.f. of $Q_1$ and $Q_2$ is given by

$$\left[\det(I - 2\theta_1B_1V - 2\theta_2B_2V)\right]^{-\frac{1}{2}}\exp\left\{\frac{1}{2}\mu^T\left[\sum_{r=1}^{\infty} 2^r(\theta_1B_1V + \theta_2B_2V)^r\right]V^{-1}\mu\right\}$$

(3) If $t = 0$ and $\theta_2 = 0$ the m.g.f. of $Q_1$ is

$$\left[\det(I - 2\theta_1B_1V)\right]^{-\frac{1}{2}}\exp\left\{\frac{1}{2}\mu^T\left[\sum_{r=1}^{\infty} 2^r(\theta_1B_1V)^r\right]V^{-1}\mu\right\}$$

Similarly the m.g.f. of $Q_2$ is given by

$$\left[\det(I - 2\theta_2B_2V)\right]^{-\frac{1}{2}}\exp\left\{\frac{1}{2}\mu^T\left[\sum_{r=1}^{\infty} 2^r(\theta_2B_2V)^r\right]V^{-1}\mu\right\}$$

(4) If $B_1VB_2 = 0$ then

$$\left[\det(I - 2\theta_1B_1V)\right]\left[\det(I - 2\theta_2B_2V)\right] = \left[\det(I - 2\theta_1B_1V - 2\theta_2B_2V)\right]
\left(\theta_1B_1V + \theta_2B_2V\right)^r = (\theta_1B_1V)^r + (\theta_2B_2V)^r$$

Thus if $B_1VB_2 = 0$ then $Q_1$ and $Q_2$ are independent.
(5) If $\mathbf{LVB} = \mathbf{0}$ then $\mathbf{W}$ and $Q = \mathbf{Y}^T \mathbf{B} \mathbf{Y}$ are independent.

(6) If $(\mathbf{BV})^2 = \mathbf{BV}$ then

$$\det(\mathbf{I} - 2\theta \mathbf{BV}) = (1 - 2\theta)^q \text{ where } q = \text{rank } (\mathbf{B}).$$

It follows that the m.g.f. of $Q$ is given by

$$M_Q(\theta) = (1 - 2\theta)^{-\frac{q}{2}} \exp \left\{ \frac{1}{2} \mu^T \left[ \sum_{r=1}^{\infty} (2\theta)^r \right] (\mathbf{BV}) \mathbf{V}^{-1} \mu \right\}$$

$$= (1 - 2\theta)^{-\frac{q}{2}} \exp \left\{ \frac{1}{2} \mu^T \left[ \frac{1}{1 - 2\theta} - 1 \right] \mathbf{B} \mu \right\}$$

$$= (1 - 2\theta)^{-\frac{q}{2}} \exp \left\{ \left[ \frac{\theta}{1 - 2\theta} \right] \mu^T \mathbf{B} \mu \right\}$$

which is the m.g.f. of a non-central chi-square distribution.

(7) If $(\mathbf{BV})^2 = \mathbf{BV}$ and $\mathbf{B} \mu = \mathbf{0}$ then the m.g.f. of $Q = \mathbf{Y}^T \mathbf{B} \mathbf{Y}$ is

$$(1 - 2\theta)^{-\frac{q}{2}}$$

so that $Q$ is chi-square with $q$ degrees of freedom.

(8) Results (4), (5), (6) and (7) are necessary and sufficient i.e. if and only if results.