Generalized extreme value distribution

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(Redirected from Extreme value distribution)

In probability theory and statistics, the **generalized extreme value distribution** (GEV) is a family of continuous probability distributions developed within extreme value theory to combine the Gumbel, Fréchet and Weibull families also known as type I, II and III extreme value distributions. Its importance arises from the fact that it is the limit distribution of the maxima of a sequence of independent and identically distributed random variables. Because of this, the GEV is used as an approximation to model the maxima of long (finite) sequences of random variables.

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### Specification

#### Probability density function

$$f(x; \mu, \sigma, \xi) = \begin{cases} \frac{1}{\sigma} (1 + \xi (z))^{-1-1/\xi} \exp\left(-\left(1+\xi z\right)^{-1/\xi}\right) & \text{for } \xi \neq 0, 
\frac{1}{\sigma} \exp\left(-\frac{x-\mu}{\sigma}\right) & \text{for } \xi = 0 \end{cases}$$

where $z = \frac{x - \mu}{\sigma}$

#### Cumulative distribution function

$$F(x; \mu, \sigma, \xi) = \begin{cases} 1 - \left(1 + \xi (z)\right)^{-1/\xi} & \text{for } \xi \neq 0, 
\exp\left(-\frac{x-\mu}{\sigma}\right) & \text{for } \xi = 0 \end{cases}$$

where $z = \frac{x - \mu}{\sigma}$

#### Mean

$$\mu - \frac{\sigma}{\xi} + \frac{\sigma}{\xi} g_1$$

where $g_k = \Gamma(1 - k\xi)$

#### Median

$$\mu + \sigma \frac{\ln^{-\xi}(2) - 1}{\xi}$$
The generalized extreme value distribution has cumulative distribution function

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<td>( \mu + \sigma \frac{(1 + \xi)^{-\xi} - 1}{\xi} )</td>
<td>( \frac{\sigma^2}{\xi^2} \left( g_2 - g_1^2 \right) )</td>
<td>( -g_3 + 3g_1g_2 - 2g_1^3 )</td>
<td>( g_4 - 4g_1g_3 + 6g_2g_1^2 - 3g_1^4 )</td>
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\[ F(x; \mu, \sigma, \xi) = \exp \left\{ - \left[ 1 + \xi \left( \frac{x - \mu}{\sigma} \right) \right]^{-1/\xi} \right\} \]

for \( 1 + \xi(x - \mu) / \sigma > 0 \), where \( \mu \in \mathbb{R} \) is the location parameter, \( \sigma > 0 \) the scale parameter and \( \xi \in \mathbb{R} \) the shape parameter.

The density function is, consequently

\[ f(x; \mu, \sigma, \xi) = \frac{1}{\sigma} \left[ 1 + \xi \left( \frac{x - \mu}{\sigma} \right) \right]^{-1/\xi - 1} \exp \left\{ - \left[ 1 + \xi \left( \frac{x - \mu}{\sigma} \right) \right]^{-1/\xi} \right\} \]

again, for \( 1 + \xi(x - \mu) / \sigma > 0 \).

**Mean, standard deviation, mode, skewness and kurtosis excess**

\[ E(X) = \mu - \frac{\sigma}{\xi} + \frac{\sigma}{\xi} g_1 \]
The skewness is

\[
\text{skewness}(X) = \frac{-g_3 + \frac{3}{2}g_1g_2 - 2g_1^3}{(g_2 - g_1^2)^{3/2}}
\]

The kurtosis excess is:

\[
\text{kurtosis excess}(X) = \frac{g_4 - 4g_1g_3 + 6g_2g_1^2 - 3g_1^4}{(g_2 - g_1^2)^2}
\]

where \( g_k = \Gamma(1-k\xi) \), \( k=1,2,3,4 \), and \( \Gamma(t) \) is Gamma function.

**Link to Fréchet, Weibull and Gumbel families**

The shape parameter \( \xi \) governs the tail behaviour of the distribution. The sub-families defined by \( \xi \rightarrow 0 \), \( \xi > 0 \) and \( \xi < 0 \) correspond, respectively, to the Gumbel, Fréchet and Weibull families, whose cumulative distribution functions are displayed below.

- **Gumbel or type I extreme value distribution**

\[
F(x; \mu, \sigma) = e^{-e^{-(x-\mu)/\sigma}} \quad \text{for} \quad x \in \mathbb{R}
\]

- **Fréchet or type II extreme value distribution**

\[
F(x; \mu, \sigma, \alpha) = \begin{cases} 
0 & x \leq \mu \\
\frac{e^{-(x-\mu)/\sigma}}{e^{-(x-\mu)/\alpha} - e^{-(\mu-\mu)/\alpha}} & x > \mu 
\end{cases}
\]

- **Reversed Weibull or type III extreme value distribution**

\[
F(x; \mu, \sigma, \alpha) = \begin{cases} 
\frac{e^{-(x-\mu)/\alpha}}{e^{-(x-\mu)/\sigma}} & x < \mu \\
1 & x \geq \mu 
\end{cases}
\]

where \( \sigma > 0 \) and \( \alpha > 0 \).
Remark I: The theory here relates to maxima and the distribution being discussed is an extreme value distribution for maxima. A Generalised Extreme Value distribution for minima can be obtained, for example by substituting \((-x)\) for \(x\) in the distribution function and this yields a separate family of distributions.

Remark II: The ordinary Weibull distribution arises in reliability applications and is obtained from the distribution here by using the variable \(t = \mu - x\), which gives a strictly positive support - in contrast to the use in the extreme value theory here. This arises because the Weibull distribution is used in cases that deal with the minimum rather than the maximum. The distribution here has an addition parameter compared to the usual form of the Weibull distribution and, in addition, is reversed so that the distribution has an upper bound rather than a lower bound. Importantly, in applications of the GEV, the upper bound is unknown and so must be estimated while when applying the Weibull distribution the lower bound is known to be zero.

Remark III: Note the differences in the ranges of interest for the three extreme value distributions: Gumbel is unlimited, Fréchet has a lower limit, while the reversed Weibull has an upper limit.

One can link the type I to types II and III the following way: if the cumulative distribution function of some random variable \(X\) is of type II: \(F(x;0,\sigma,a)\), then the cumulative distribution function of \(\ln X\) is of type I, namely \(F(x;\ln \sigma,1/a)\). Similarly, if the cumulative distribution function of \(X\) is of type III: \(F(x;0,\sigma,a)\), the cumulative distribution function of \(\ln X\) is of type I: \(F(x; -\ln \sigma,1/a)\).

**Extremal types theorem**

Credit for the extremal types theorem (or convergence to types theorem) is given to Gnedenko (1948), previous versions were stated by Fisher and Tippett in 1928 and Fréchet in 1927.

Let \(X_1, X_2 \ldots\) be a sequence of independent and identically distributed random variables, let \(M_n = \max\{X_1, \ldots, X_n\}\). If two sequences of real numbers \(a_n, b_n\) exist such that \(a_n > 0\) and

\[
\lim_{n \to \infty} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = F(x)
\]

then if \(F\) is a non degenerate distribution function, it belongs to either the Gumbel, the Fréchet or the Weibull family.

Clearly, the theorem can be reformulated saying that \(F\) is a member of the GEV family.
It is worth noting that the result, which is stated for maxima, can be applied to minima by taking the sequence \( -X_n \) instead of the sequence \( X_n \).

For the practical application this theorem means: For samples taken from a well behaving, arbitrary distribution \( X \) the resulting extreme value distribution \( M_n \) can be approximated and parametrised with the extreme value distribution with the appropriate support.

Thus the role of extremal types theorem for maxima is similar to that of central limit theorem for averages. The latter states that the limit distribution of arithmetic mean of a sequence \( X_n \) of random variable is the normal distribution no matter what the distribution of the \( X_n \). The extremal types theorem is similar in scope where maxima is substituted for average and GEV distribution is substituted for normal distribution.

**References**


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