3 Random Vectors

3.1 Definition: A random vector is a vector of random variables $\mathbf{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$.

3.2 Definition: The mean or expectation of $\mathbf{X}$ is defined as $E[\mathbf{X}] = \begin{pmatrix} E[X_1] \\ \vdots \\ E[X_n] \end{pmatrix}$.

3.3 Definition: A random matrix is a matrix of random variables $\mathbf{Z} = (Z_{ij})$. Its expectation is given by $E[\mathbf{Z}] = (E[Z_{ij}])$.

3.4 Theorem: A constant vector $\mathbf{a}$ (vector of constants) and a constant matrix $\mathbf{A}$ (matrix of constants) satisfy $E[\mathbf{a}] = \mathbf{a}$ and $E[\mathbf{A}] = \mathbf{A}$.

3.5 Theorem: $E[\mathbf{X} + \mathbf{Y}] = E[\mathbf{X}] + E[\mathbf{Y}]$.

3.6 Theorem: $E[\mathbf{AX}] = \mathbf{A}E[\mathbf{X}]$ for a constant matrix $\mathbf{A}$.

3.7 Theorem: $E[\mathbf{AZB} + \mathbf{C}] = \mathbf{A}E[\mathbf{Z}]\mathbf{B} + \mathbf{C}$ if $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are constant matrices.

3.8 Definition: If $\mathbf{X}$ is a random vector, the covariance matrix of $\mathbf{X}$ is defined as

$$\text{cov}(\mathbf{X}) \equiv [\text{cov}(X_i, X_j)] \equiv \begin{pmatrix} \text{var}(X_1) & \text{cov}(X_1, X_2) & \cdots & \text{cov}(X_1, X_n) \\ \text{cov}(X_2, X_1) & \text{var}(X_2) & \cdots & \text{cov}(X_2, X_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{cov}(X_n, X_1) & \text{cov}(X_n, X_2) & \cdots & \text{var}(X_n) \end{pmatrix}.$$

An alternative form is

$$\text{cov}(\mathbf{X}) = E[(\mathbf{X} - E[\mathbf{X}])(\mathbf{X} - E[\mathbf{X}])'] = E\left[\begin{pmatrix} X_1 - E[X_1] \\ \vdots \\ X_n - E[X_n] \end{pmatrix} (X_1 - E[X_1], \cdots, X_n - E[X_n])\right].$$

3.9 Example: If $X_1, \ldots, X_n$ are independent, then the covariances are 0 and the covariance matrix is equal to $\text{diag}(\sigma_1^2, \ldots, \sigma_n^2)$, or $\sigma^2\mathbf{I}_n$ if the X_i have common variance σ^2.

Properties of covariance matrices:

3.10 Theorem: Symmetry: $\text{cov}(\mathbf{X}) = [\text{cov}(\mathbf{X})]'$.

3.11 Theorem: $\text{cov}(\mathbf{X} + \mathbf{a}) = \text{cov}(\mathbf{X})$ if $\mathbf{a}$ is a constant vector.

3.12 Theorem: $\text{cov}(\mathbf{AX}) = \mathbf{A}\text{cov}(\mathbf{X})\mathbf{A}'$ if $\mathbf{A}$ is a constant matrix.

3.13 Theorem: $\text{cov}(\mathbf{X})$ is p.s.d.

3.14 Theorem: $\text{cov}(\mathbf{X})$ is p.d. provided no linear combination of the X_i is a constant.

3.15 Theorem: $\text{cov}(\mathbf{X}) = E[\mathbf{X}\mathbf{X}'] - E[\mathbf{X}](E[\mathbf{X}])'$

3.16 Definition: The correlation matrix of $\mathbf{X}$ is defined as

$$\text{corr}(\mathbf{X}) = [\text{corr}(X_i, X_j)] \equiv \begin{pmatrix} 1 & \text{corr}(X_1, X_2) & \cdots & \text{corr}(X_1, X_n) \\ \text{corr}(X_2, X_1) & 1 & \cdots & \text{corr}(X_2, X_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{corr}(X_n, X_1) & \text{corr}(X_n, X_2) & \cdots & 1 \end{pmatrix}.$$

3.17 Note: Denote $\text{cov}(\mathbf{X})$ by $\mathbf{\Sigma} = (\sigma_{ij})$. Then the correlation matrix and covariance matrix are related by

$$\text{cov}(\mathbf{X}) = \text{diag}(\sqrt{\sigma_{11}}, \ldots, \sqrt{\sigma_{nn}}) \times \text{corr}(\mathbf{X}) \times \text{diag}(\sqrt{\sigma_{11}}, \ldots, \sqrt{\sigma_{nn}}).$$

This is easily seen using $\text{corr}(X_i, X_j) = \text{cov}(X_i, X_j)/\sqrt{\sigma_{ii}\sigma_{jj}}$.

3.18 Example: If $X_1, \ldots, X_n$ are exchangeable, they have a constant variance σ^2 and a constant correlation ρ between any pair of variables. Thus

$$\text{cov}(\mathbf{X}) = \sigma^2 \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{pmatrix}.$$

This is sometimes called an exchangeable covariance matrix.

3.19 Definition: If $\mathbf{X}_{m\times 1}$ and $\mathbf{Y}_{n\times 1}$ are random vectors,

$$\text{cov}(\mathbf{X}, \mathbf{Y}) = [\text{cov}(X_i, Y_j)] \equiv \begin{pmatrix} \text{cov}(X_1, Y_1) & \text{cov}(X_1, Y_2) & \cdots & \text{cov}(X_1, Y_n) \\ \text{cov}(X_2, Y_1) & \text{cov}(X_2, Y_2) & \cdots & \text{cov}(X_2, Y_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{cov}(X_m, Y_1) & \text{cov}(X_m, Y_2) & \cdots & \text{cov}(X_m, Y_n) \end{pmatrix}.$$

An alternative form is:

$$\text{cov}(\mathbf{X}, \mathbf{Y}) = E[(\mathbf{X} - E[\mathbf{X}])(\mathbf{Y} - E[\mathbf{Y}])'] = E\left[\begin{pmatrix} X_1 - E[X_1] \\ \vdots \\ X_m - E[X_m] \end{pmatrix} (Y_1 - E[Y_1], \cdots, Y_n - E[Y_n])\right].$$

3.20 Theorem: If $\mathbf{A}$ and $\mathbf{B}$ are constant matrices, then $\mathrm{cov}(\mathbf{AX}, \mathbf{BY}) = \mathbf{A}\,\mathrm{cov}(\mathbf{X}, \mathbf{Y})\,\mathbf{B}'$.

3.21 Theorem: Let $\mathbf{Z} = \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix}$. Then $\mathrm{cov}(\mathbf{Z}) = \begin{pmatrix} \mathrm{cov}(\mathbf{X}) & \mathrm{cov}(\mathbf{X}, \mathbf{Y}) \\ \mathrm{cov}(\mathbf{Y}, \mathbf{X}) & \mathrm{cov}(\mathbf{Y}) \end{pmatrix}$.

3.22 Theorem: Let $E[\mathbf{X}] = \boldsymbol{\mu}$ and $\mathrm{cov}(\mathbf{X}) = \boldsymbol{\Sigma}$ and $\mathbf{A}$ be a constant matrix. Then

$$E[(\mathbf{X} - \boldsymbol{\mu})'\mathbf{A}(\mathbf{X} - \boldsymbol{\mu})] = \mathrm{tr}(\mathbf{A}\boldsymbol{\Sigma}).$$

3.23 Theorem: $E[\mathbf{X}'\mathbf{AX}] = \mathrm{tr}(\mathbf{A}\boldsymbol{\Sigma}) + \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu}$.

3.24 Example: Let $X_1, \ldots, X_n$ be independent random variables with common mean μ and variance σ^2. Then the sample variance $S^2 = \sum_i (X_i - \bar{X})^2/(n-1)$ is an unbiased estimate of σ^2.

3.25 Theorem: If $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\mathbf{A}(= \mathbf{A}')$ and $\mathbf{B}$ are constant matrices, then $\mathbf{X}'\mathbf{AX}$ and $\mathbf{BX}$ are independently distributed iff $\mathbf{B}\boldsymbol{\Sigma}\mathbf{A} = \mathbf{0}$.

3.26 Example: Let $X_1, \ldots, X_n$ be independent normal random variables with common mean μ and variance σ^2. Then the sample mean $\bar{X} = \sum_{i=1}^n X_i/n$ and the sample variance S^2 are independently distributed.