A Probabilistic Causal Model for Mediation with Interference

Guanglei Hong
University of Chicago

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A Probabilistic Causal Model for Defining the Average Treatment Effect

Unstable Unit Treatment Value

Philosophical Question:

- To be or not to be if treated…; or if untreated…
- Statistics appreciates uncertainties in the world rather than simply assuming them away

A Probabilistic Causal Model

- For unit i assigned to treatment a, the potential outcome $Z_i(a)$ has a probability distribution.

- Let $Z_i(a) = 1$ if the patient survives and 0 otherwise, then

$$Z_i(a) \sim \text{Bernoulli}\ (p_i(a))$$

- The individual-specific treatment effect:

$$\Delta_i = p_i(1) - p_i(0)$$

- For $i \neq j$, we may have that $Z_i(a) = Z_j(a)$ when $p_i(a) \neq p_j(a)$; or that $Z_i(a) \neq Z_j(a)$ when $p_i(a) = p_j(a)$.

Population Model

- Causal Question

  Does the intervention change the population average survival rate?

$$\delta = E\{\Delta\} = E\{p(1) - p(0)\} = E\{E[Z(1) - Z(0)]\}$$

Sub-Population Model

- In a subpopulation defined by pretreatment covariates $\mathbf{X} = \mathbf{x}$, individual units have *independent* and *identical* distributions of potential outcomes under a given treatment a.

- The subpopulation-specific average treatment effect

$$\delta_{\mathbf{x}} = E\{\Delta \mid \mathbf{X} = \mathbf{x}\} = p_{\mathbf{x}}(1) - p_{\mathbf{x}}(0)$$

Here $p_{\mathbf{x}}(a) = pr(Z(a) = 1 \mid \mathbf{X} = \mathbf{x})$ for $a = 0, 1$.

A Probabilistic Causal Model for Defining the Mediated Treatment Effect

Application Example 1

- Causal Question: Is an intervention (A)'s effect on the final outcome (Y) mediated by an intermediate outcome (Z)?
- Intervention: Random assignment to the new welfare-to-work policy ($A = 1$) or the old welfare system ($A = 0$)
- Outcome: Depression (Y) at the two-year follow-up; $Y = 1$ if depressed and 0 otherwise
- Mediator: Employment experience during the two years after randomization; $Z = 1$ if ever employed and 0 otherwise

Potential Outcome as a Function of Treatment and Mediator

- Welfare recipient *i*'s potential mediator value under treatment *a* is subject to the random fluctuation in the local job market:

$$Z_i(a) \sim \text{Bernoulli}\ (p_i(a))$$

- Welfare recipient *i*'s potential outcome value under treatment *a* and mediator value $Z_i(a) = z$ is also subject to random external influences:

$$Y_i(a, z) \sim \text{Bernoulli}\ (q_i(a, z))$$

Treatment Effect Decomposition

- Extending Pearl (2001) and Robins & Greenland (1992), for welfare recipient *i*,

$$\Delta_i = E\{Y_i(1, Z_i(1)) - Y_i(0, Z_i(0))\}$$

$$= \underbrace{E\{Y_i(1, Z_i(0)) - Y_i(0, Z_i(0))\}} - \underbrace{E\{Y_i(1, Z_i(1)) - Y_i(1, Z_i(0))\}}$$

$\Delta_i^{(\mathrm{NDE})}$: Natural Direct Effect $\Delta_i^{(\mathrm{NIE})}$: Natural Indirect Effect

Unit-Specific Distributions of Potential Outcomes

- $E\{Y_i(1, Z_i(1))\} = q_i(1, 1) \times p_i(1) + q_i(1, 0) \times [1 - p_i(1)];$
- $E\{Y_i(0, Z_i(0))\} = q_i(0, 1) \times p_i(0) + q_i(0, 0) \times [1 - p_i(0)];$
- $E\{Y_i(1, Z_i(0))\} = q_i(1, 1) \times p_i(0) + q_i(1, 0) \times [1 - p_i(0)].$

Population Average Effects

- Population average natural direct effect:

$$\delta^{(\mathrm{NDE})} = E\{\Delta^{(\mathrm{NDE})}\} = E\{E[Y(1, Z(0)) - Y(0, Z(0))]\}$$

- Population average natural indirect effect:

$$\delta^{(\mathrm{NIE})} = E\{\Delta^{(\mathrm{NIE})}\} = E\{E[Y(1, Z(1)) - Y(1, Z(0))]\}$$

- Population average potential outcomes:

$$E\{E[Y(0, Z(0))]\}, E\{E[Y(1, Z(1))]\}, E\{E[Y(1, Z(0))]\}$$

Hypothetical Sequential Randomized experiment

- Welfare recipients are assigned at random to the intervention or the control condition

1. $q(1, 1), q(1, 0), q(0, 1), q(0, 0) \perp A$

2. $p(1), p(0) \perp A$

- Under each treatment condition, welfare recipients are assigned at random to either employment or unemployment

3. $p(1) > 0; p(0) > 0$

4. $q(1, 1), q(1, 0) \perp Z(1) \mid A = 1; \quad q(0, 1), q(0, 0) \perp Z(0) \mid A = 0$

5. $q(1, 1), q(1, 0) \perp Z(0) \mid A = 1$

Hypothetical Sequential Randomized experiment

$E\{E[Y(0,Z(0))]\}$

$1-p(0)=.6$ $E[q(0,0)]=.2$
$p(0)=.4$ $E[q(0,1)]=.2$

$E\{E[Y(0,Z(0))]\}$
$=E\{Y \mid A=0\}$
$=.2$

$E\{E[Y(1,Z(1))]\}$

$1-p(1)=.3$ $E[q(1,0)]=.4$
$p(1)=.7$ $E[q(1,1)]=.2$

$E\{E[Y(1,Z(1))]\}$
$=E(Y \mid A=1)$
$=.26$

$\omega=\frac{0.6}{0.3}=2$

$\omega=\frac{0.4}{0.7}=\frac{4}{7}$

$E\{E[Y(1,Z(0))]\}$

$1-p(0)=.6$ $E[q(1,0)]=.4$
$p(0)=.4$ $E[q(1,1)]=.2$

$E\{E[Y(1,Z(0))]\}$
$=E(\omega Y \mid A=1)$
$=.32$

Subpopulation Average Potential Outcomes

- Within a subpopulation defined by covariates $\mathbf{X} = \mathbf{x}$, let

$$p_{\mathbf{x}}(a) = pr(Z(a) = 1 \mid \mathbf{X} = \mathbf{x})$$

$$q_{\mathbf{x}}(a,z) = pr(Y(a,z) = 1 \mid \mathbf{X} = \mathbf{x})$$

Subpopulation Average Potential Outcomes

- $E\{E[Y(1, Z(1)] \mid \mathbf{X} = \mathbf{x}\}$
  $= q_{\mathbf{x}}(1, 1) \times p_{\mathbf{x}}(1) + q_{\mathbf{x}}(1, 0) \times [1 - p_{\mathbf{x}}(1)]$
- $E\{E[Y(0, Z(0)] \mid \mathbf{X} = \mathbf{x}\}$
  $= q_{\mathbf{x}}(0, 1) \times p_{\mathbf{x}}(0) + q_{\mathbf{x}}(0, 0) \times [1 - p_{\mathbf{x}}(0)]$
- $E\{E[Y(1, Z(0)] \mid \mathbf{X} = \mathbf{x}\}$
  $= q_{\mathbf{x}}(1, 1) \times p_{\mathbf{x}}(0) + q_{\mathbf{x}}(1, 0) \times [1 - p_{\mathbf{x}}(0)]$
  $= E\{E[\omega Y(1, Z(1)] \mid \mathbf{X} = \mathbf{x}\}$

where

$\omega = p_{\mathbf{x}}(0) / p_{\mathbf{x}}(1)$ when $A = 1, Z(1) = 1$;

$\omega = [1 - p_{\mathbf{x}}(0)] / [1 - p_{\mathbf{x}}(1)]$ when $A = 1, Z(1) = 0$

Identification Assumptions

Within a subpopulation defined by covariates $\mathbf{X} = \mathbf{x}$,

for $a = 0, 1$ and $z = 0, 1$

1. $q_{\mathbf{x}}(a, \mathrm{z}) \perp A$
2. $p_{\mathbf{x}}(a) \perp A$
3. $p_{\mathbf{x}}(1) > 0$ when $p_{\mathbf{x}}(0) > 0$
4. $q_{\mathbf{x}}(1, z) \perp Z(1) \mid A = 1$; $q_{\mathbf{x}}(0, z) \perp Z(0) \mid A = 0$
5. $q_{\mathbf{x}}(1, z) \perp Z(0) \mid A = 1$

Ratio-of-Mediator-Probability Weighting

RMPW for units assigned to the intervention $A = 1$ and displaying mediator value z

$$\omega = pr(Z(0) = z \mid A = 0, \mathbf{X} = \mathbf{x}) \,/\, pr(Z(1) = z \mid A = 1, \mathbf{X} = \mathbf{x})$$

Here the numerator and the denominator of the weight can each be estimated through analyzing a propensity score model.

Outcome Model

- Merge the original sample with a duplicate set of the experimental group

Treatment (A)	Duplicate (D)	RMPW	Estimate
0	0	1	$E\{E[Y(0, Z(0)]\}$
1	0	ω	$E\{E[Y(1, Z(0)]\}$
1	1	1	$E\{E[Y(1, Z(1)]\}$

- Weighted outcome model (with robust standard errors):

$$Y = \gamma + \delta^{(NDE)} A + \delta^{(NIE)} D + e$$

Relative Strengths of RMPW

- The RMPW method does not assume
  - No treatment-by-mediator interaction
  - Exclusion restriction
  - Functional form of the outcome model

Simulation Results

In estimating population average NDE and NIE,

- Parametric weighting removes at least 98% of the initial bias; non-parametric weighting with 4×4 strata removes at least 87% of the initial bias.
- Non-parametric weighting estimates are relatively more efficient than parametric weighting estimates; hence their MSEs are comparable in most cases.
- The discrepancy between robust SE estimates and sampling variability is close to zero
- The weighting estimates replicate path analysis and IV results when the assumption of no treatment-by-mediator interaction or the exclusion restriction holds; the weighting method outperforms when these assumptions do not hold.

A Probabilistic Causal Model for Mediation with Interference

Application Example 2

- Causal Question: Is an intervention (A)'s effect on the final outcome (Y) mediated by interference between units (Z)?
- Intervention: A policy requiring all ninth graders to take algebra ($A = 1$) versus not requiring algebra ($A = 0$)
- Outcome: Math achievement (Y) at the end of the ninth grade
- Mediator: Class peer ability (Z) which depends on whether the school reorganizes math classes in response to the new policy

Potential Mediators and Potential Outcomes

- For lower-achieving students who would take remedial math in the absence of the algebra requirement,

  $Z(1) > Z(0)$ if the school creates mixed-ability algebra classes

  $Z(1) = Z(0)$ if the school continues to sort students into math classes by ability when offering algebra

  Other random factors may contribute to class peer ability:

  $$Z_i(a) \sim \text{Normal}\ (\mu_i(a), \sigma_{z(a)}{}^2)$$

  Random factors may influence individual math outcome:

  $$Y_i(a, z) \sim \text{Normal}\ (\upsilon_i(a, z), \sigma_{y(a,z)}{}^2)$$

RMPW for Normally Distributed Mediator

- To estimate the population average NDE and NIE,

$$E\{E[Y(1, Z(0))] \mid \mathbf{X} = \mathbf{x}\} = E\{E[\omega Y(1, Z(1))] \mid \mathbf{X} = \mathbf{x}\}$$

where, for a student who has been exposed to the intervention and has experienced class peer ability level z,

$$\omega = \frac{\sigma_{Z(1)}}{\sigma_{Z(0)}} \exp\left[\frac{\left(z - \mu_{(1)}\right)^2}{2\sigma^2_{Z(1)}} - \frac{\left(z - \mu_{(0)}\right)^2}{2\sigma^2_{Z(0)}}\right]$$

Estimate $\mu_{(1)}, \sigma^2_{Z(1)}, \mu_{(0)},$ and $\sigma^2_{Z(0)}$ as functions of $\mathbf{X}$.

RMPW for Ordinally Distributed Mediator

For a student who has been exposed to the intervention and has experienced class peer ability level z,

$$\omega = \frac{pr\left(Z(0) = z \mid A = 0, \mathbf{X}\right)}{pr\left(Z(1) = z \mid A = 1, \mathbf{X}\right)}$$

Evidence for the Spillover Effect

For students who would experience, due to the policy,

- A rise in class peer ability *and*
- An increase in algebra enrollment

- The total effect was indistinguishable from zero (coefficient = 0.23, *SE* = 1.15, *t* = 0.20)

- Negative indirect effect of the policy (coefficient = 2.70, *SE* = 1.20, *t* = 2.24)
  - A rise in class peer ability may put low-ability students at a disadvantage possibly due to unfavorable social comparisons or due to instruction beyond reach

- Positive direct effect of the policy (coefficient = -2.33, *SE* = 0.88, *t* = -2.63)
  - Taking algebra may benefit low-ability student's learning had their class peer ability remained unchanged

Related Articles

- Hong, G., Deutsch, J., & Hill, H. (Under review). Parametric and non-parametric weighting methods for estimating mediation effects: An application to the National Evaluation of Welfare-to-Work Strategies.

- Hong, G., & Nomi, T. (2012). Weighting Methods for Assessing Policy Effects Mediated by Peer Change. To appear in the *Journal of Research on Educational Effectiveness* special issue on the statistical approaches to studying mediator effects in education research.