Using Mixed Integer Programming for Matching in Observational Studies

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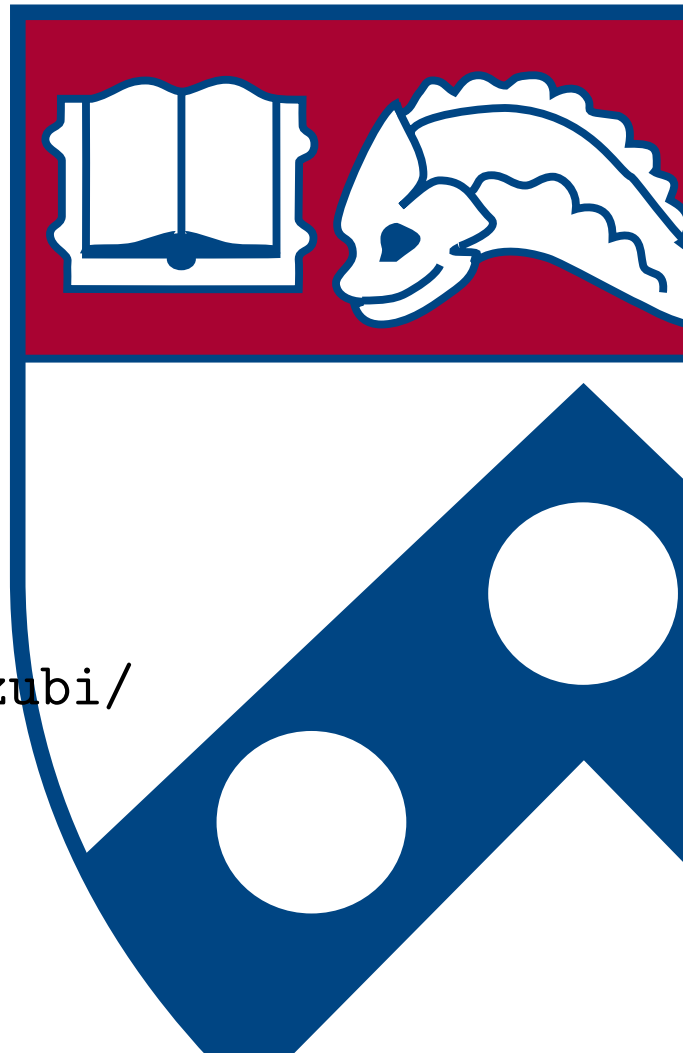

http://www-stat.wharton.upenn.edu/~josezubi/

Key takeaway points

- Optimal matching method
  - Get the balance you want
  - Know it is infeasible
    - Eliminate guesswork
- Directly balance several statistics beyond means
- Keep the adjustments simple enough
  - People can talk about them
  - Sensitivity analysis to unobserved biases

Outline

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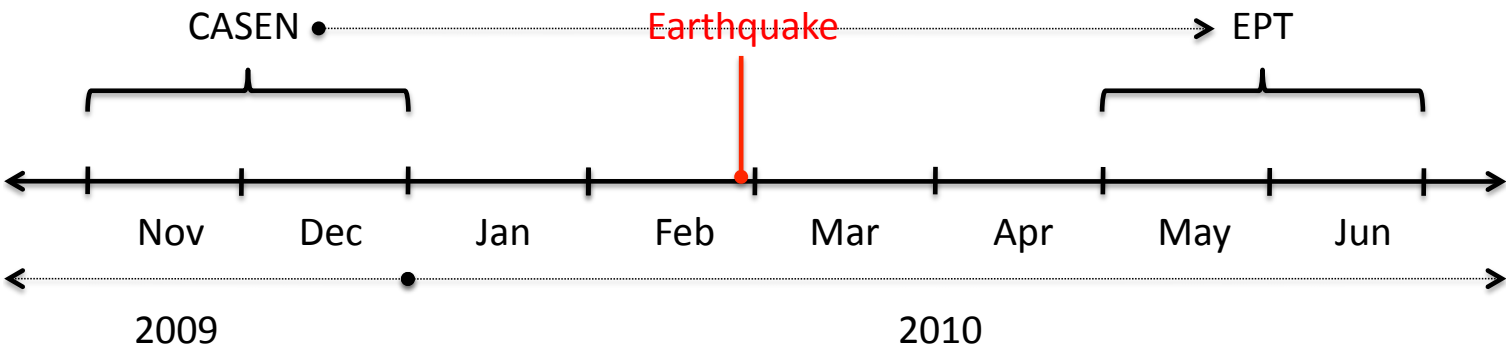
The 2010 Chilean earthquake

4th strongest earthquake in the world in the last 50 years

Sebastián Martínez/AP Photo

Effect of the earthquake

- Effect of the earthquake on posttraumatic stress?
- The post earthquake survey (EPT)

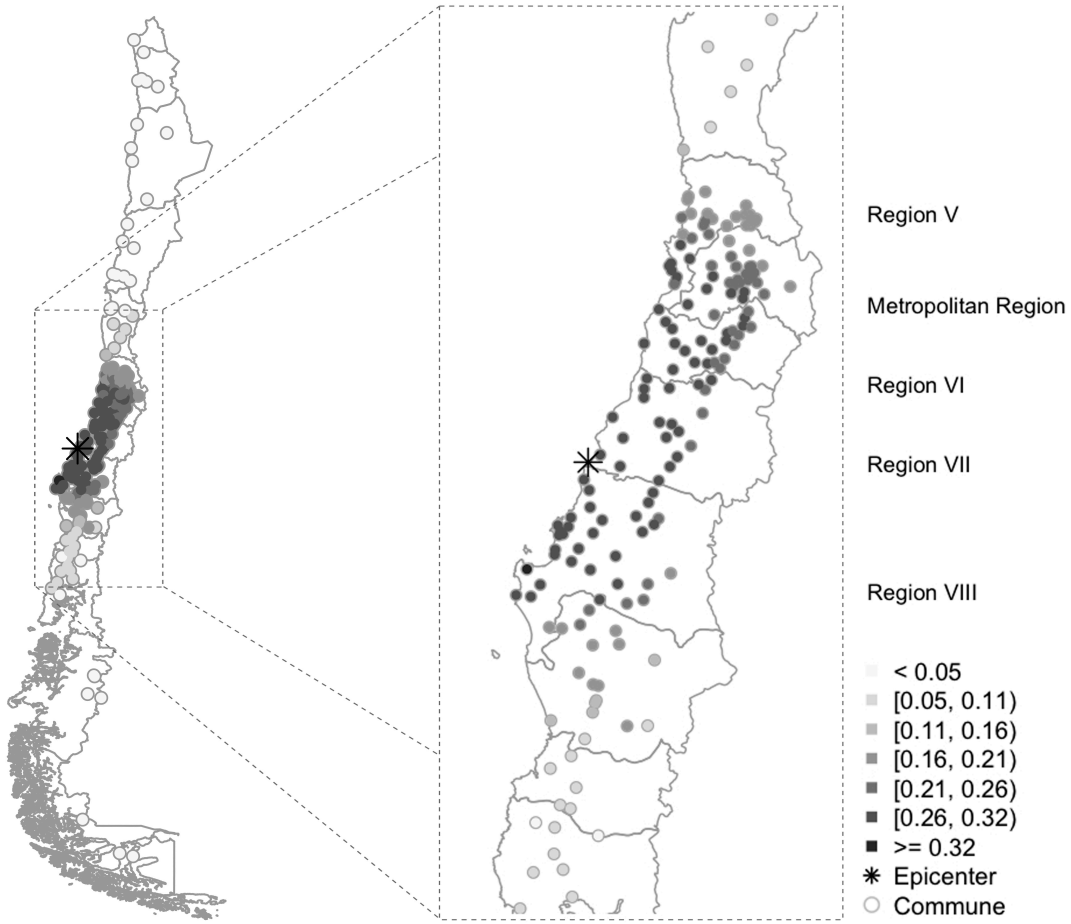


Re-interviewed 22,456 households from CASEN 2009

Detailed measurements of the **same** individuals before and after

Intensity of the earthquake

- Peak ground acceleration (PGA) in the communes of the EPT



Matched design

- Matched respondents with PGA < 0.014 to those with PGA ≥ 0.275
- We matched
  - exactly for sex, age and ethnic groups
  - with fine balance for self-rated health, quality of the housing
  - balancing the entire empirical distributions of income
  - mean balancing the 46 covariates in the study

Outline

Notation

- Let $\mathcal{T} = \{t_1, ..., t_T\}$ be the set of treated units, and $\mathcal{C} = \{c_1, ..., c_C\}$, the set of potential controls, with $T \leq C$
- Define $\mathcal{P} = \{p_1, ..., p_P\}$ as the set of observed covariates
- Each treated unit $t \in \mathcal{T}$ has a vector of observed covariates $\mathbf{x}_{t,\cdot} = \{\mathbf{x}_{t,p_1}, ..., \mathbf{x}_{t,p_P}\}$, and each control $c \in \mathcal{C}$ has a similar vector $\mathbf{x}_{c,\cdot} = \{\mathbf{x}_{c,p_1}, ..., \mathbf{x}_{c,p_P}\}$
- Based on these covariates there is a distance $\mathbf{0} \leq \delta_{t,c} < \infty$ between treated and control units
- Decision variable

$$a_{t,c} = \begin{cases} 1 & \text{if treated } t \text{ is assigned to control } c \\ 0 & \text{otherwise} \end{cases}$$

The assignment algorithm

$$\begin{aligned}
\underset{\mathbf{a}}{\text{minimize}} \quad & \sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} \delta_{t,c} a_{t,c} \\
\text{subject to} \quad & \sum_{c \in \mathcal{C}} a_{t,c} = m, \ \ t \in \mathcal{T} \\
& \sum_{t \in \mathcal{T}} a_{t,c} \leq 1, \ \ c \in \mathcal{C} \\
& a_{t,c} \in \{0, 1\}, \ \ t \in \mathcal{T}, c \in \mathcal{C}
\end{aligned}$$

A MIP with direct balance via the objective function

$$\underset{\mathbf{a}}{\text{minimize}} \quad \sum_{t\in\mathcal{T}}\sum_{c\in\mathcal{C}} \delta_{t,c}a_{t,c} + \sum_{j\in\mathcal{J}} \omega_i\mu_i(\mathbf{a})$$

$$\text{subject to} \quad \sum_{c\in\mathcal{C}} a_{t,c} = m, \;\; t \in \mathcal{T}$$

$$\sum_{t\in\mathcal{T}} a_{t,c} \leq 1, \;\; c \in \mathcal{C}$$

$$a_{t,c} \in \{0,1\}, \;\; t \in \mathcal{T}, c \in \mathcal{C}$$

A MIP with direct balance via the constraints

$$\underset{\mathbf{a}}{\text{minimize}} \quad \sum_{t\in\mathcal{T}}\sum_{c\in\mathcal{C}} \delta_{t,c} a_{t,c}$$

$$\text{subject to} \quad \sum_{c\in\mathcal{C}} a_{t,c} = m, \;\; t \in \mathcal{T}$$

$$\sum_{t\in\mathcal{T}} a_{t,c} \leq 1, \;\; c \in \mathcal{C}$$

$$a_{t,c} \in \{0,1\}, \;\; t \in \mathcal{T}, c \in \mathcal{C}$$

$$\nu_j(\mathbf{a}) \leq \varepsilon_j, \; j \in \mathcal{J}$$

Outline

Balancing the means of the covariates (1)

$$\underset{\mathbf{a}}{\text{minimize}} \quad \sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} \delta_{t,c} a_{t,c} + \sum_{j \in \mathcal{J}} \omega_j \left| \sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} \frac{x_{c,j} a_{t,c}}{mT} - \overline{x}_{\mathcal{T},j} \right|$$

$$\text{subject to} \quad \sum_{c \in \mathcal{C}} a_{t,c} = m, \;\; t \in \mathcal{T}$$

$$\sum_{t \in \mathcal{T}} a_{t,c} \leq 1, \;\; c \in \mathcal{C}$$

$$a_{t,c} \in \{0, 1\}, \;\; t \in \mathcal{T}, c \in \mathcal{C}$$

Balancing the means of the covariates (1)

$$\underset{\mathbf{a},\mathbf{z}}{\text{minimize}} \quad \sum_{t\in\mathcal{T}}\sum_{c\in\mathcal{C}} \delta_{t,c} a_{t,c} + \sum_{j\in\mathcal{J}} \omega_j z_j$$

$$\text{subject to} \quad \sum_{c\in\mathcal{C}} a_{t,c} = m, \;\; t \in \mathcal{T}$$

$$\sum_{t\in\mathcal{T}} a_{t,c} \leq 1, \;\; c \in \mathcal{C}$$

$$a_{t,c} \in \{0,1\}, \;\; t \in \mathcal{T}, c \in \mathcal{C}$$

$$z_j \geq \sum_{t\in\mathcal{T}}\sum_{c\in\mathcal{C}} \frac{x_{c,j} a_{t,c}}{mT} - \overline{x}_{\mathcal{T},j}, \; j \in \mathcal{J}$$

$$z_j \geq -\sum_{t\in\mathcal{T}}\sum_{c\in\mathcal{C}} \frac{x_{c,j} a_{t,c}}{mT} + \overline{x}_{\mathcal{T},j}, \; j \in \mathcal{J}$$

Balancing the means of the covariates (1)

$$\underset{\mathbf{a},\mathbf{z}}{\text{minimize}} \quad \sum_{t\in\mathcal{T}}\sum_{c\in\mathcal{C}} \delta_{t,c} a_{t,c}$$

$$\text{subject to} \quad \sum_{c\in\mathcal{C}} a_{t,c} = m, \;\; t \in \mathcal{T}$$

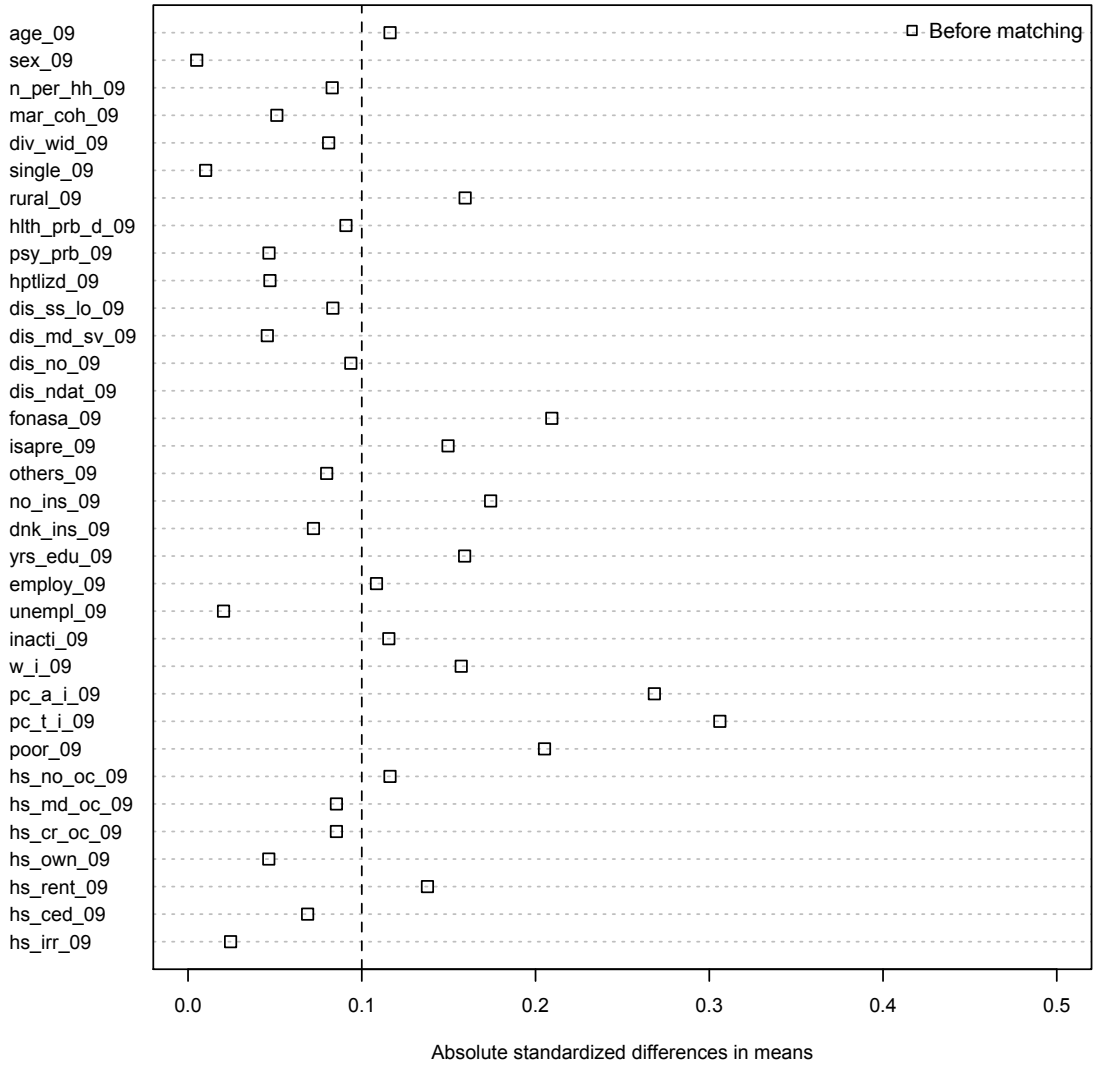
$$\sum_{t\in\mathcal{T}} a_{t,c} \leq 1, \;\; c \in \mathcal{C}$$

$$a_{t,c} \in \{0,1\}, \;\; t \in \mathcal{T}, c \in \mathcal{C}$$

$$\sum_{t\in\mathcal{T}}\sum_{c\in\mathcal{C}} \frac{x_{c,j} a_{t,c}}{mT} - \overline{x}_{\mathcal{T},j} \leq \varepsilon_j, \; j \in \mathcal{J}$$

$$-\sum_{t\in\mathcal{T}}\sum_{c\in\mathcal{C}} \frac{x_{c,j} a_{t,c}}{mT} + \overline{x}_{\mathcal{T},j} \leq \varepsilon_j, \; j \in \mathcal{J}$$

Balancing the means of the covariates (2)



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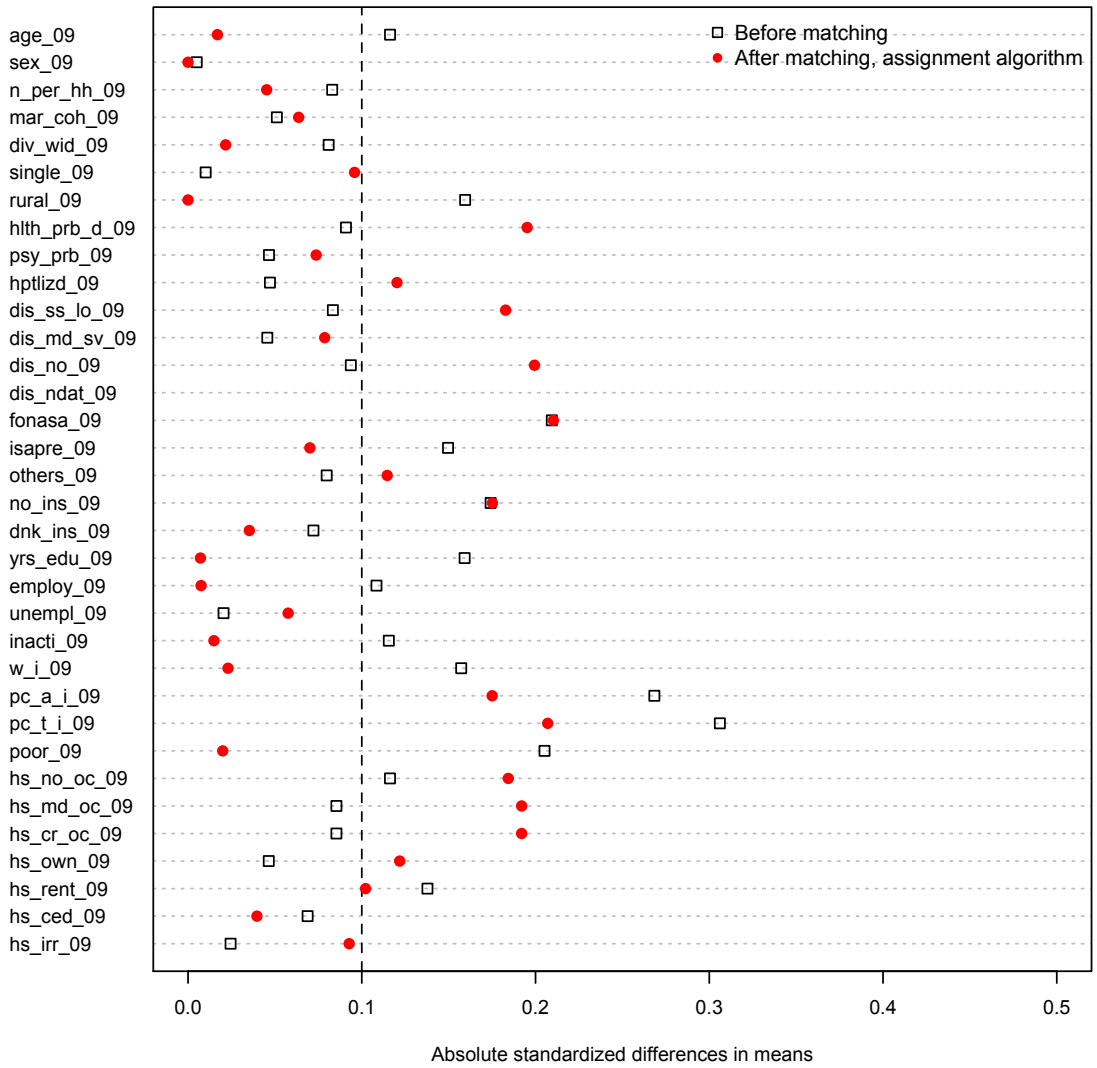

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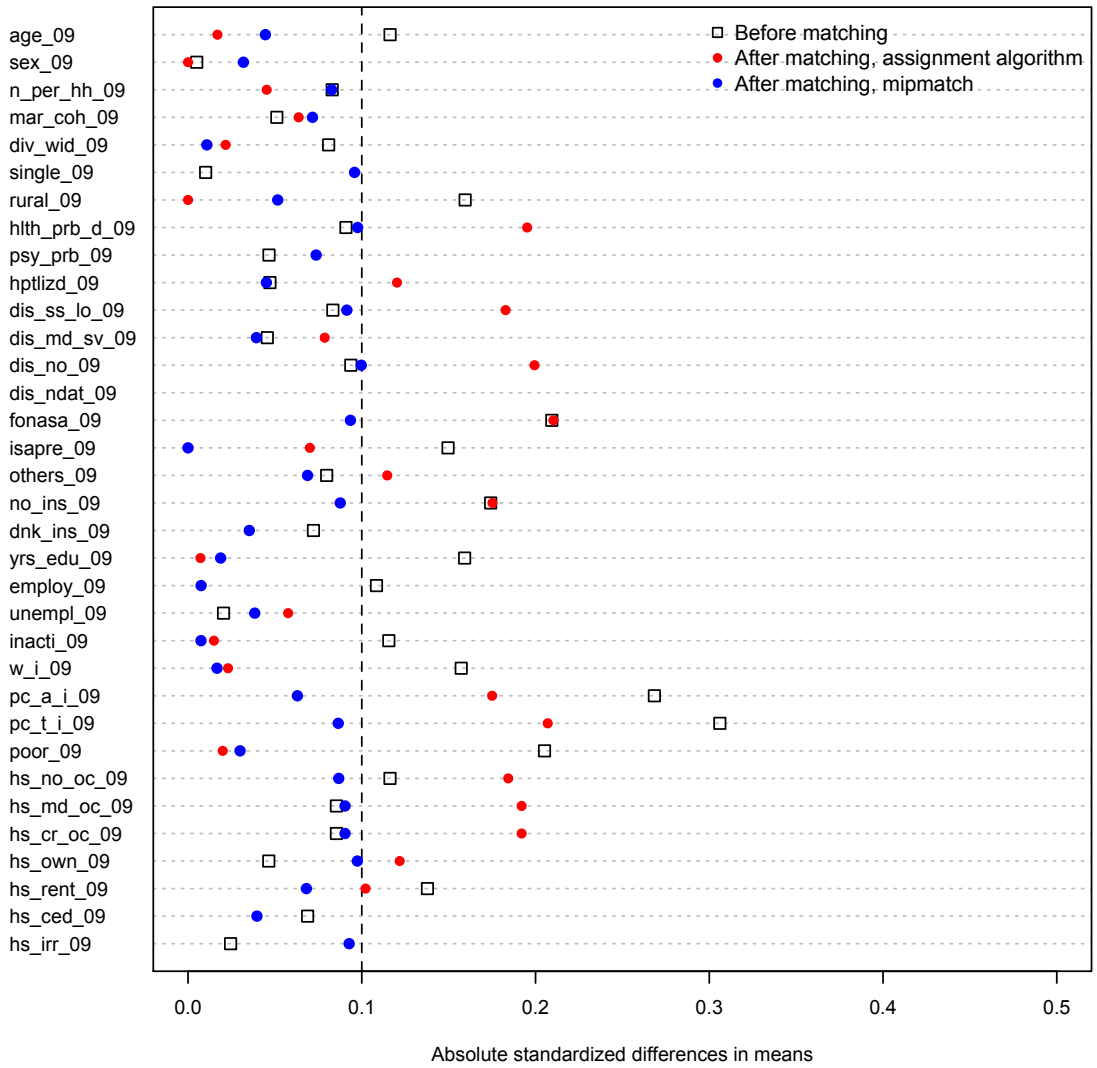

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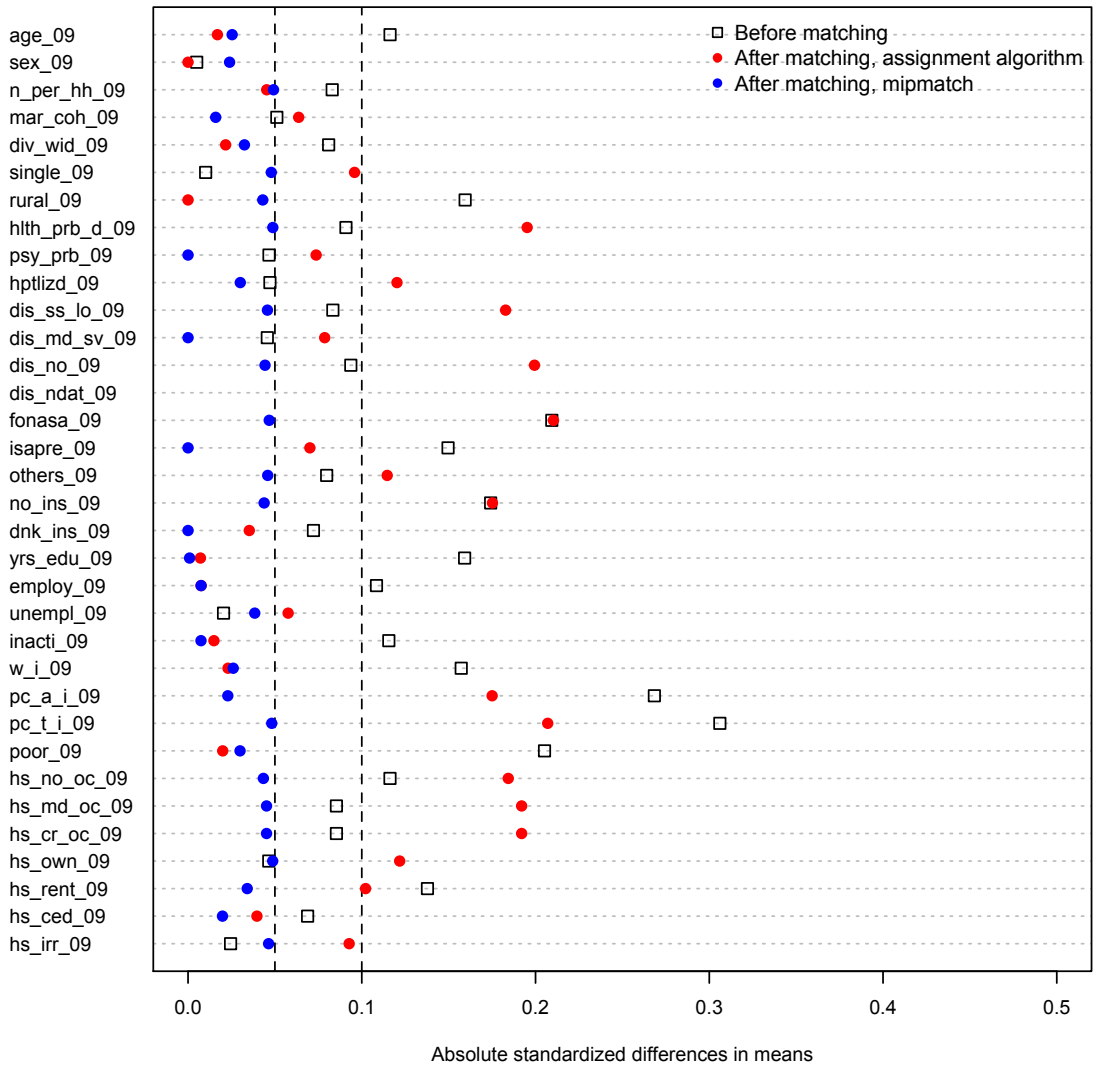

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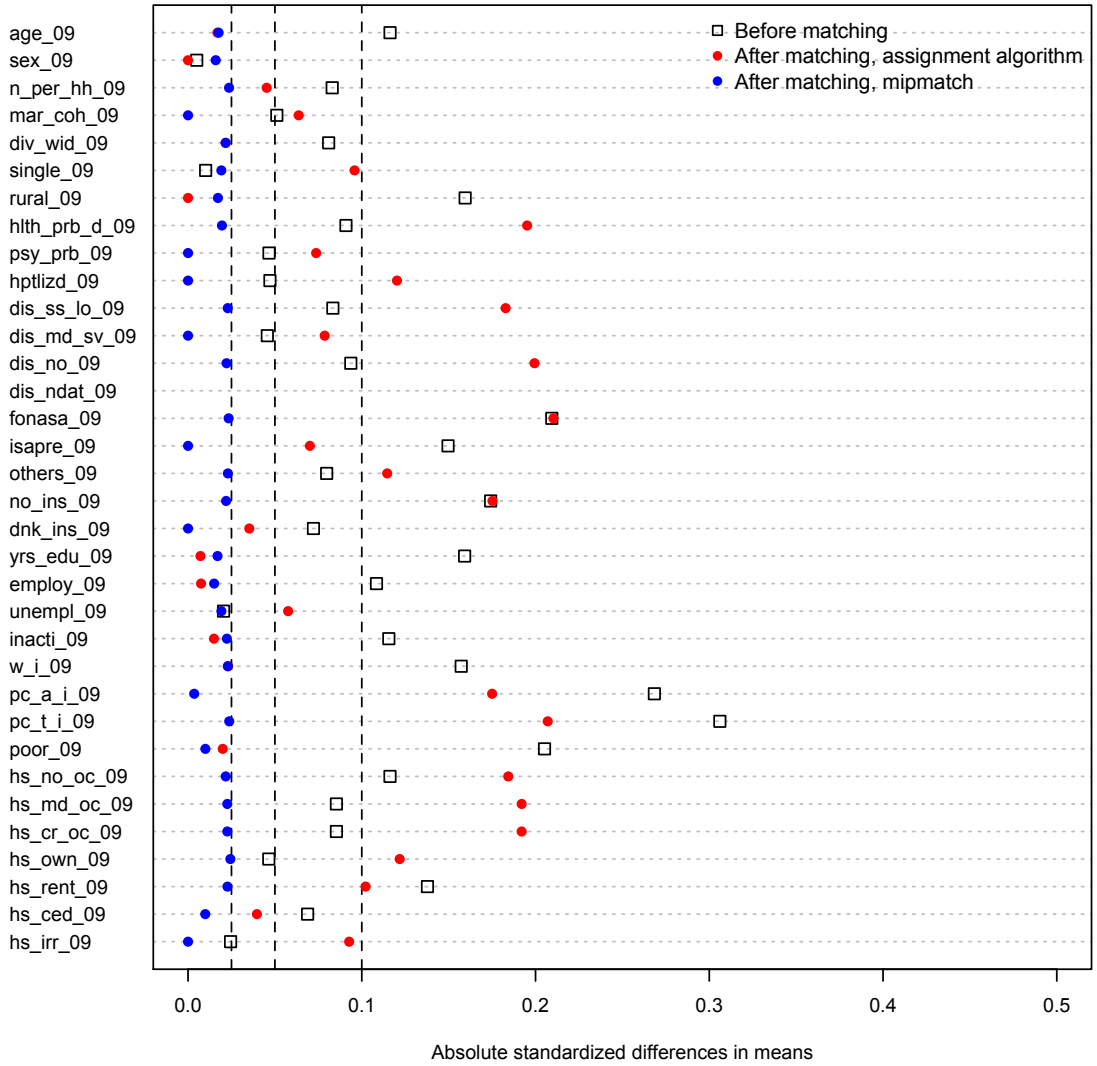


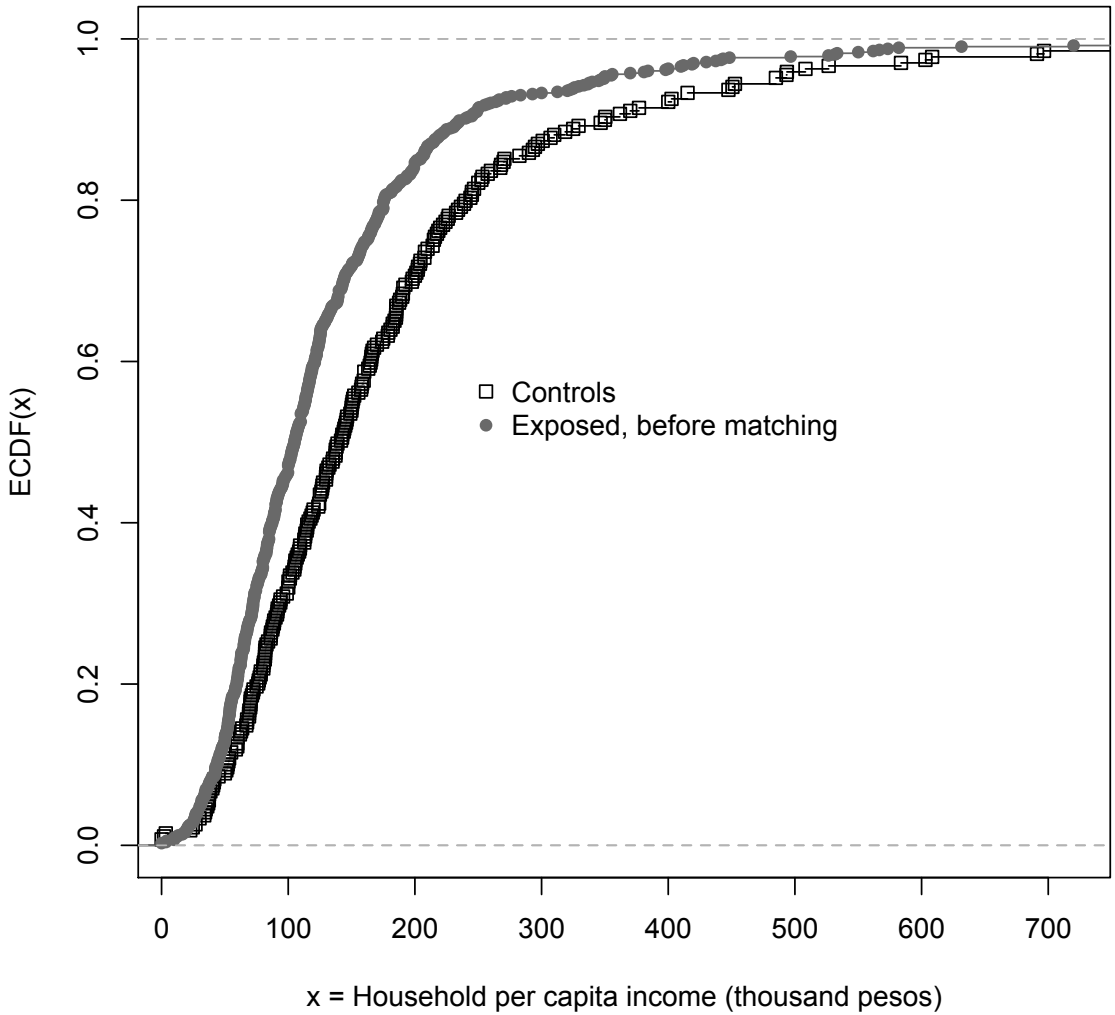
Balancing Kolmogorov-Smirnov statistics (1)

$$\sum_{j \in \mathcal{J}} \omega_j \mu_j(\mathbf{a}) = \omega_j \sup_{x_{c,p} \in \mathcal{G}(\mathbf{x}_{\mathcal{T},p})} \left| F_{\mathcal{T}}(x_{c,p}) - F_{\mathcal{C}}(x_{c,p}, \mathbf{a}) \right|$$

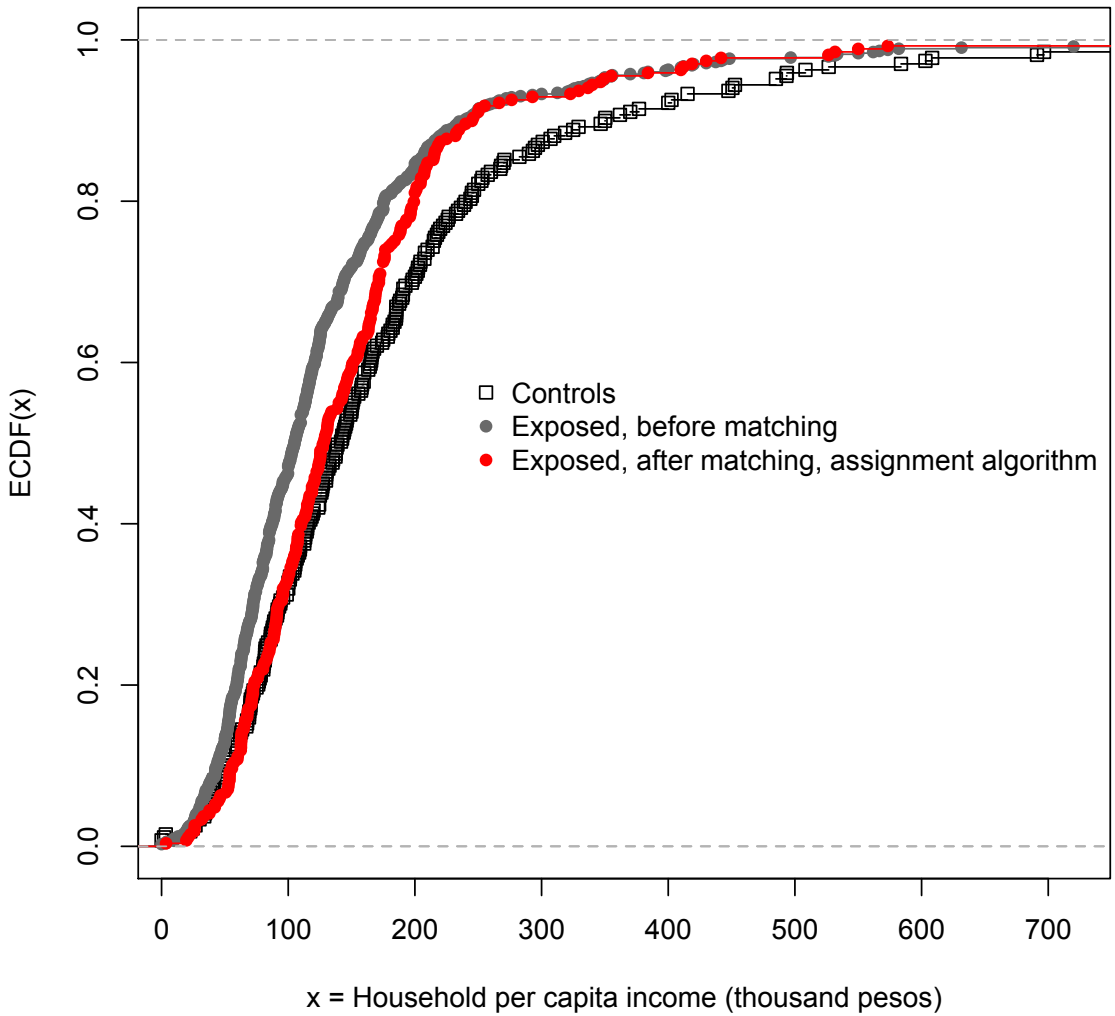
$$= \omega_j z_j$$

$$\geq \left| \frac{1}{|\mathcal{G}(\mathbf{x}_{\mathcal{T},p})|} - \sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} \frac{\mathbb{1}_{\{x_{g-1;p} \leq x_{c,p} < x_{g;p}\}} a_{t,c}}{mT} \right| \quad \forall x_{g;p} \in \mathcal{G}(\mathbf{x}_{\mathcal{T},p})$$

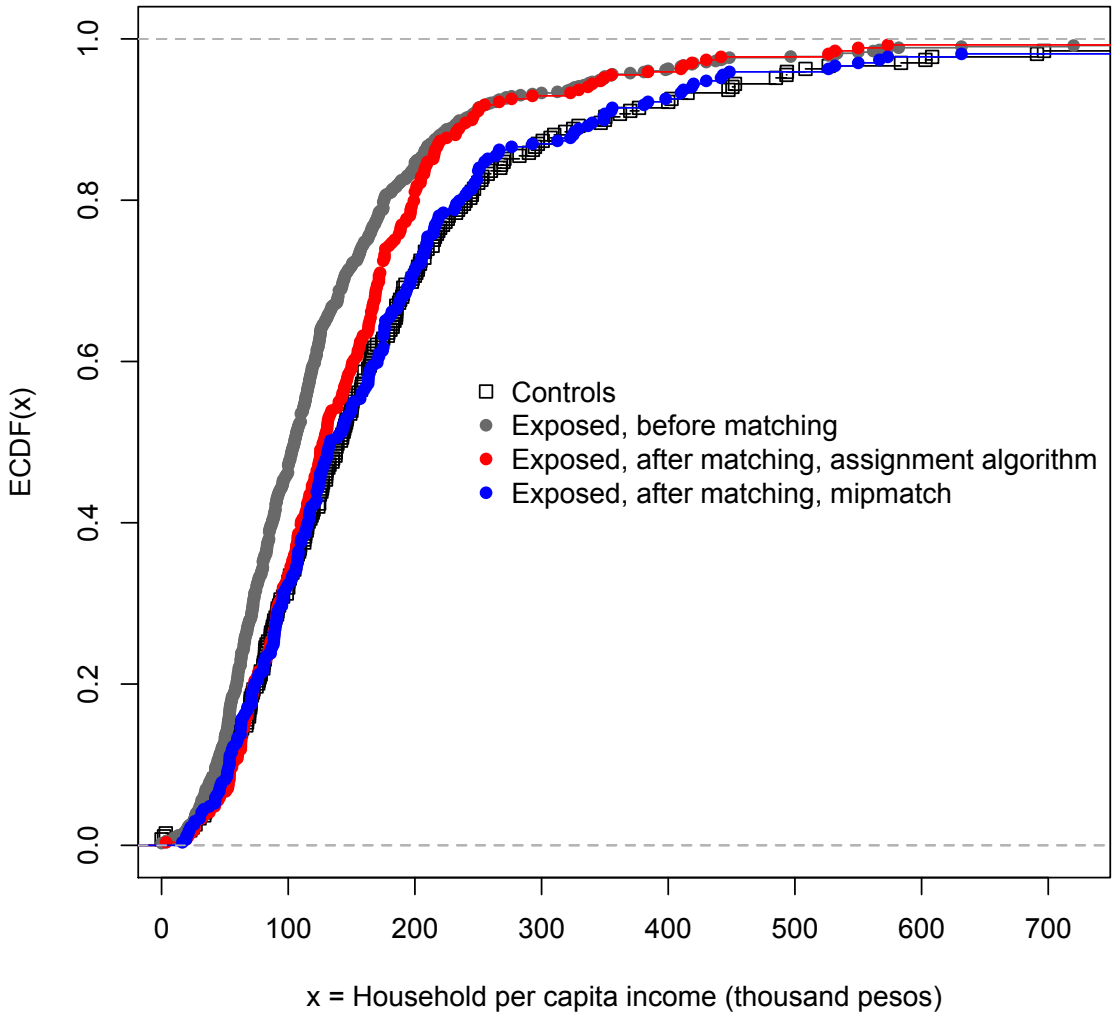
Balancing Kolmogorov-Smirnov statistics (2)



Balancing Kolmogorov-Smirnov statistics (2)



Balancing Kolmogorov-Smirnov statistics (2)



Fine and near-fine balance for several covariates (1)

Fine balance:

$$\sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} a_{t,c} \mathbb{1}_{\{x_{c,p}=b\}} = m \sum_{t \in \mathcal{T}} \mathbb{1}_{\{x_{t,p}=b\}} \qquad \forall b \in \mathcal{B}$$

Near-fine balance:

$$\left| \sum_{t \in \mathcal{T}} \sum_{c \in \mathcal{C}} a_{t,c} \mathbb{1}_{\{x_{c,p}=b\}} - m \sum_{t \in \mathcal{T}} \mathbb{1}_{\{x_{t,p}=b\}} \right| \leq \xi \qquad \forall b \in \mathcal{B}$$

Fine and near-fine balance for several covariates (2)

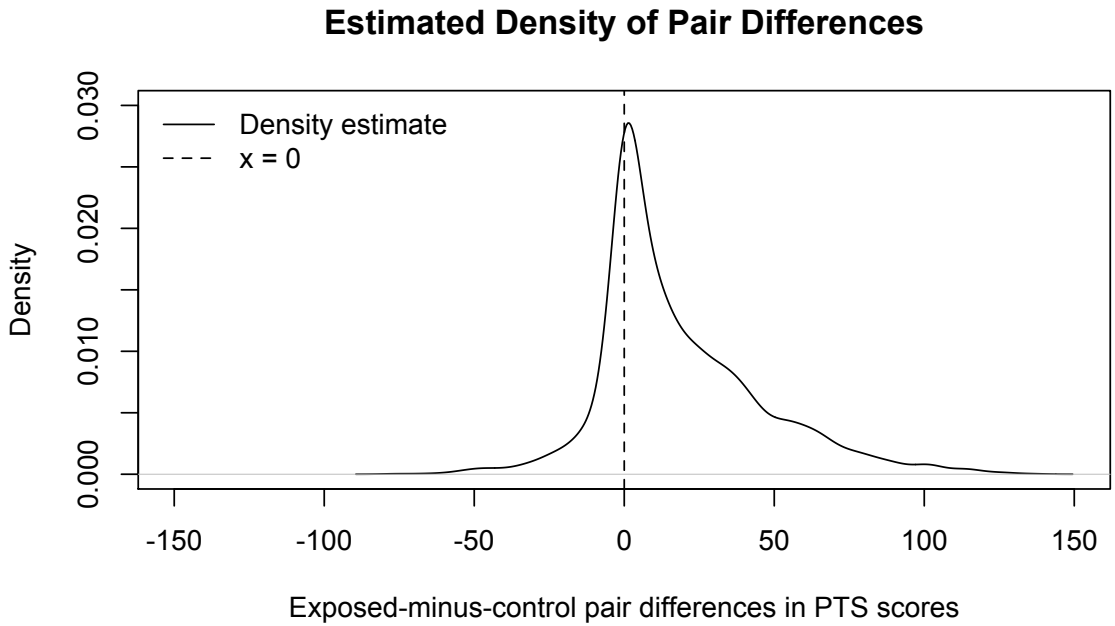
Table: Fine balance for self-rated health

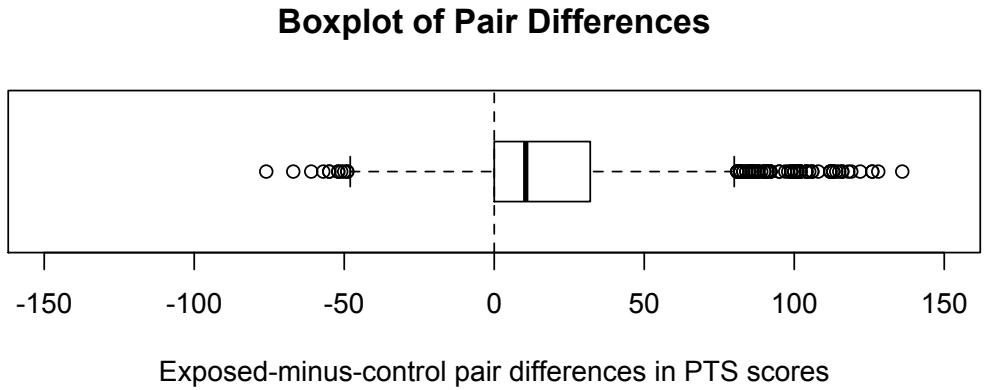
	Exposed	Controls
Poor	122	122
Good	1487	1487
Fair	911	911

Table: Fine balance for material quality of the housing

	Exposed	Controls
Acceptable	1738	1738
Unacceptable	739	739
Beyond repair	43	43

Density plot of PTS scores





Outline

Summary and remarks

- Explicitly optimize or constrain the criteria used to assess the quality of the match
  - Meet the criteria
  - Know that the criteria is infeasible
- Directly balance
  - Means
  - Variances and skewness
  - Correlations
  - Quantiles
  - Kolmogorov-Smirnov statistic
- While matching with exact, near-exact, fine and near-fine balance for more than one covariate
- A systematic method for improving covariate balance

Extensions

- Optimal subset matching
- Building a stronger instrumental variable
- Enhancing regression discontinuity designs
- `R` package `mipmatch`

References

- Zubizarreta, J. R. (2012), "Using Mixed Integer Programming for Matching in an Observational Study of Acute Kidney Injury after Surgery," under revision.
- Zubizarreta, J. R., Cerdá, M. and Rosenbaum, P. R. (2012), "Effect of the 2010 Chilean Earthquake on Posttraumatic Stress: Designing an Observational Study to be Less Sensitive to Unmeasured Biases," under revision.
- Zubizarreta, J. R., Reinke, C. E., Kelz, R. R., Silber, J. H. and Rosenbaum, P. R. (2011), "Matching for Several Sparse Nominal Variables in a Case-Control Study of Readmission Following Surgery," *The American Statistician*, 65, 229-238.

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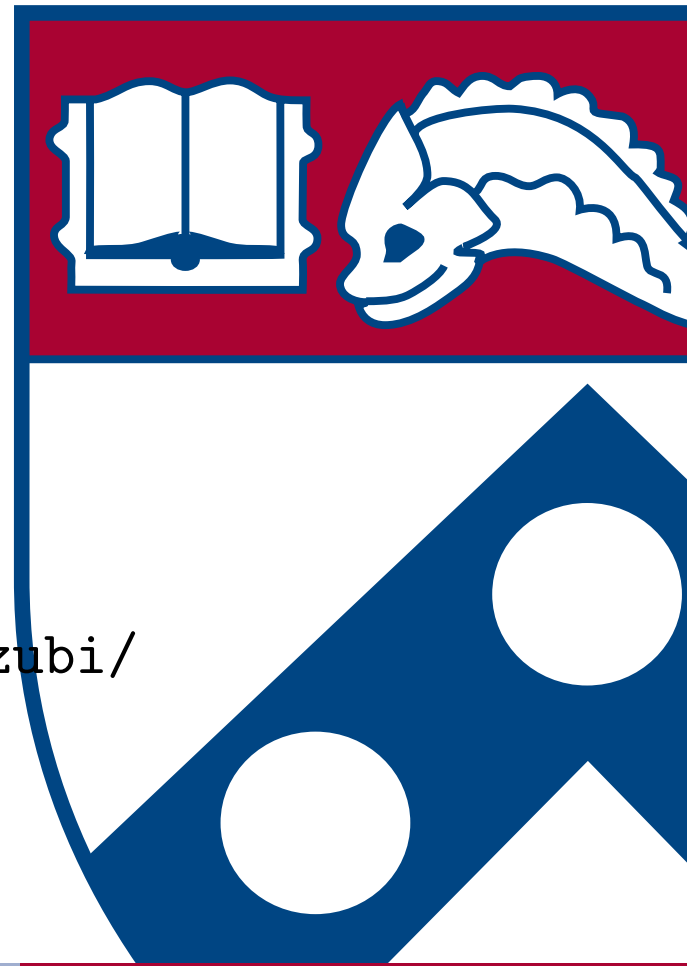

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