Homework Assignment #1
(Due Wednesday, September 28, 2005)

Please hand in a hard copy of your R code in addition to your solutions and plots. In addition, please send an electronic version of your code to Kenny (kshum@jhsph.edu).

1. Show the following:

   (a) If the distribution of Y belongs to the exponential family, the moment generating function of Y is

   $$M_Y(t) = \exp\left\{\frac{b(a(\phi)t + \theta) - b(\theta)}{a(\phi)}\right\}.$$

   (b) The binomial distribution defined by

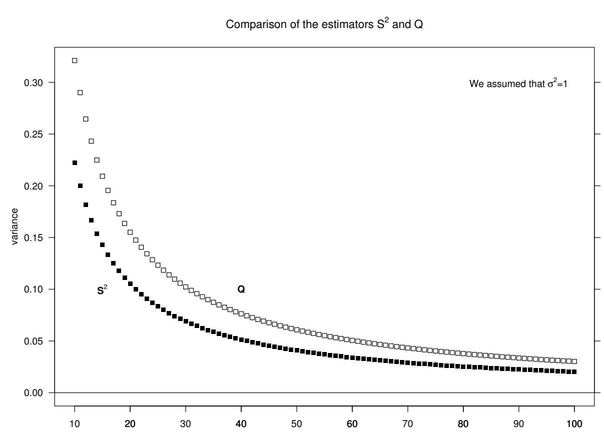
   $$f(y_i, n_i, p_i) = \binom{n_i}{y_i} p_i^{y_i} (1 - p_i)^{n_i - y_i}$$

   belongs to the exponential family. Also verify its mean and variance relationship.

2. Let $X_1, X_2, \ldots, X_n$ be independently and identically distributed as $N(\mu, \sigma^2)$. Define

   $$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 \qquad \text{and} \qquad Q = \frac{1}{2(n-1)}\sum_{i=1}^{n-1}(X_{i+1} - X_i)^2.$$

   (a) Prove that $\text{var}(S^2) = 2\sigma^4/(n-1)$.
   (b) Show Q is an unbiased estimator of σ^2.
   (c) Find the variance of Q and hence show that, as $n \to \infty$, the effeciency of Q relative to S^2 is $\frac{2}{3}$.
   (d) Use R to plot the variances S^2 and Q as functions of n (with n between 10 and 100).



Hint: If $\mathbf{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then $\text{var}(\mathbf{Y}'\mathbf{A}\mathbf{Y}) = 2 \times \text{tr}(\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}) + 4\boldsymbol{\mu}'\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\mu}$ (see for example Searle, p57).

3. Let $A = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}$.

   (a) Show that $A^n \to \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$ as $n \to \infty$.

   (b) Write an iteration in R for the above. Stop after step k if $\max_{i,j} |a_{ij}^k - a_{ij}^{k-1}| < 10^{-6}$. How many iterations does it take?

4. Suppose

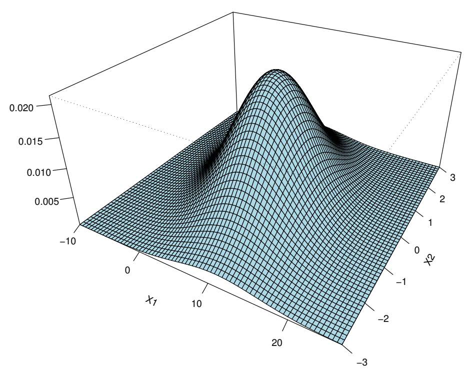
$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} \sim N_4\left(\begin{bmatrix} 3 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \right) = N_4(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

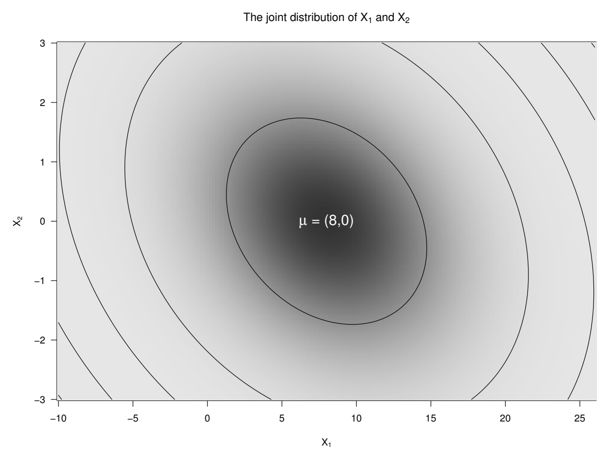
   Let $X_1 = Y_1 + Y_2 + Y_3 + Y_4$, and $X_2 = Y_1 - Y_2 - Y_3 + Y_4$.

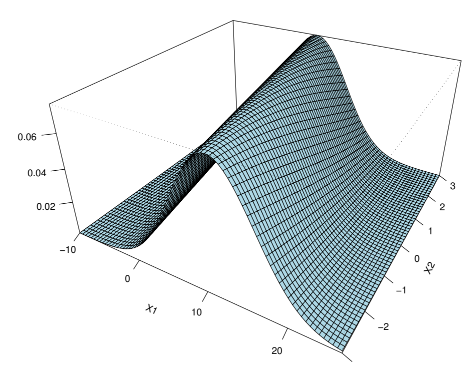
   (a) Find the joint distribution of $(X_1, X_2)'$.

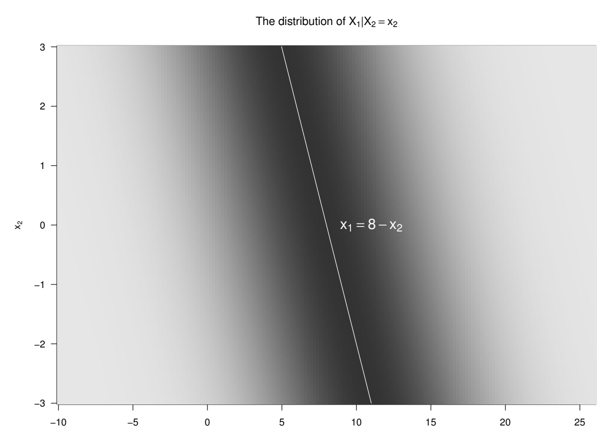
   (b) Find the conditional distribution of X_1 given X_2.

   (c) For the above distributions, draw 3D density plots and contour plots using R.









Functions that you might find useful include apply, contour, expand.grid, expression, image, lines, persp, plot, plotmath, text.