Advanced Methods in Biostatistics 1 (140.751)
Fall 2005

Homework Assignment #4
(Due Monday, October 24, 2005)

Please hand in a hard copy of your R code, and send an electronic version of it to Kenny (kshum@jhsph.edu).

1. Let A_1, A_2 and A_3 be objects of unknown weights β_1, β_2 and β_3 respectively. To obtain the weights of A_1, A_2 and A_3, we use the following three weighings on a balance (scale), all of which are repeated twice.

   - A_1 on the balance (results y_{11} and y_{12}),
   - A_2 on the balance (results y_{21} and y_{22}),
   - A_3 on the balance (results y_{31} and y_{32}).

   Assume that the y_{ij}'s come from random variables Y_{ij} which are independent, normally distributed with the same variance σ^2. Also assume that the balance has an unknown systematic error θ.

   (a) Show that the β_i's are not estimable.

   (b) Show that $\beta_i - \beta_j$ for $i \neq j$ are estimable. Find the BLUE of $\beta_1 - \beta_2$ and an estimate of its variance.

   (c) If we include two more readings in which we weigh all three objects on the balance (results y_{41} and y_{42}), show that the β_i's are estimable and that θ is estimable. Find the BLUE of each of the estimates, and an estimate of the standard error of each estimate.

2. (a) If $H : \mathbf{A}\boldsymbol{\beta} = \mathbf{0}$ is a testable hypothesis, show that

   $$\text{cov}(\mathbf{A}\hat{\boldsymbol{\beta}}) = \sigma^2 \mathbf{A}(\mathbf{X}'\mathbf{X})^- \mathbf{A}'.$$

   (b) Assuming rank$(\mathbf{A}) = q$, show that

   $$(RSS_H - RSS)/\sigma^2 = (\mathbf{A}\hat{\boldsymbol{\beta}})'[\text{cov}(\mathbf{A}\hat{\boldsymbol{\beta}})]^{-1}(\mathbf{A}\hat{\boldsymbol{\beta}}).$$

   (c) Conclude from the above that

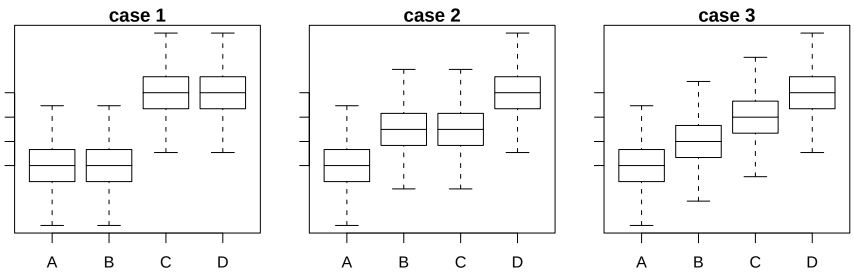
   $$E[RSS_H - RSS] = \sigma^2 q + (\mathbf{A}\boldsymbol{\beta})'[\mathbf{A}(\mathbf{X}'\mathbf{X})^- \mathbf{A}']^{-1}(\mathbf{A}\boldsymbol{\beta}),$$

   and when $H : \mathbf{A}\boldsymbol{\beta} = \mathbf{0}$ is true and $\mathbf{Y} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2 \mathbf{I})$,

   $$(RSS_H - RSS)/\sigma^2 \sim \chi^2_q.$$

   In the above exercise, you can use the fact that dim$(\omega^\perp \cap \Omega) = $ dim$(\mathcal{R}(\mathbf{P}_\Omega \mathbf{M}')) = q$ if rank $(\mathbf{M}) = q$.

3. Consider a one-way ANOVA with 4 groups. Assume there are n observations per group. We are testing the hypothesis that all group means are equal, and the usual assumptions are met. Assume the following cases, in which the null is false:



In case 1, the group means of A and B are equal, the group means of C and D are equal, but the group means of A and D differ by 3 units ($3c$, say). In case 2, the group means of B and C are equal, but the group means of A and B differ by $1.5c$, and the group means of C and D differ by $1.5c$. In case 3, the group means of A and B differ by $1c$, the group means of B and C differ by $1c$, and the group means of C and D differ by $1c$.

(a) For each of those cases, calculate the non-centrality parameter λ for the distribution in the F-test as a function of c/σ, where σ^2 is the within-group variance.

(b) For each of those cases, plot the power of the F-test as a function of c/σ, for $n = 5, 10, 15, 20$.

(c) For $n = 10$ and cases 1 and 2, verify your findings by a simulation.