Confidence Intervals for Proportions

Example

[Carroll, *J Med Entomol* **38**:114–117, 2001]

Place tick on clay island surrounded by water, with two capillary tubes: one treated with deer-gland-substance; one untreated.

Tick sex	Leg	Deer sex	treated	untreated
male	fore	female	24	5
female	fore	female	18	5
male	fore	male	23	4
female	fore	male	20	4
male	hind	female	17	8
female	hind	female	25	3
male	hind	male	21	6
female	hind	male	25	2

$\longrightarrow$ Is the tick more likely to go to the treated tube?

Test for a proportion

Suppose X $\sim$ Binomial(n, p).

Test $H_0 : p = \frac{1}{2}$ vs $H_a : p \neq \frac{1}{2}$.

Reject H_0 if $X \geq H$ or $X \leq L$.

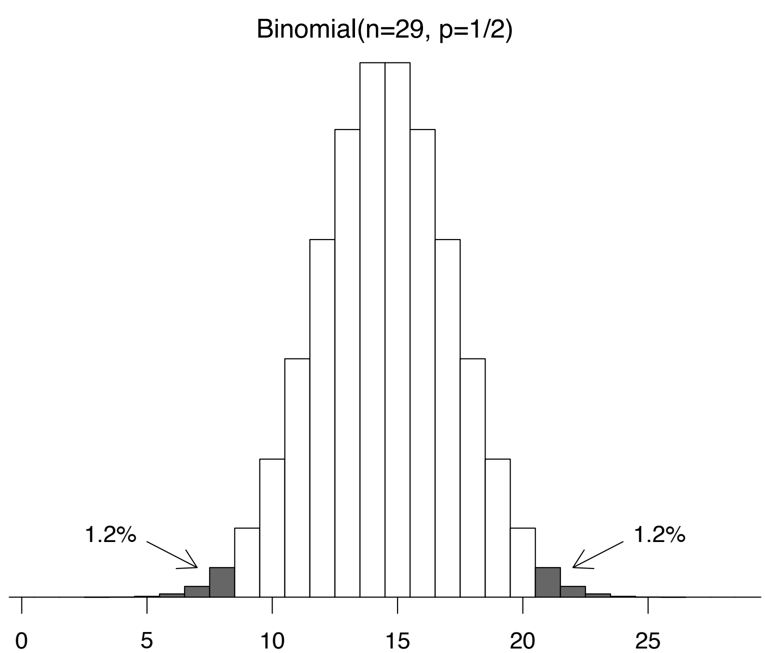
Choose H and L such that

$$\Pr(X \geq H \mid p = \tfrac{1}{2}) \leq \alpha/2 \quad \text{and} \quad \Pr(X \leq L \mid p = \tfrac{1}{2}) \leq \alpha/2.$$

Thus $\Pr(\text{Reject } H_0 \mid H_0 \text{ is true}) \leq \alpha$.

$\longrightarrow$ The difficulty: The Binomial distribution is hard to work with. Because of its discrete nature, you can't get exactly your desired significance level (α).

Binomial(n=29, p=1/2)



Rejection region

Consider $X \sim$ Binomial(n=29, p).

Test of $H_0 : p = \frac{1}{2}$ vs $H_a : p \neq \frac{1}{2}$ at significance level α = 0.05.

Lower critical value:

`qbinom(0.025, 29, 0.5) = 9`

$\Pr(X \leq 9)$ = `pbinom(9, 29, 0.5) = 0.031` $\rightarrow$ L = 8

Upper critical value:

`qbinom(0.975, 29, 0.5) = 20`

$\Pr(X \geq 20)$ = `1-pbinom(19,29,0.5) = 0.031` $\rightarrow$ H = 21

Reject H_0 if $X \leq 8$ or $X \geq 21$. (For testing $H_0 : p = \frac{1}{2}$, H = n – L)

Significance level

Consider $X \sim$ Binomial(n=29, p).

Test of $H_0 : p = \frac{1}{2}$ vs $H_a : p \neq \frac{1}{2}$ at significance level α = 0.05.

Reject H_0 if $X \leq 8$ or $X \geq 21$.

Actual significance level:

$$\begin{aligned}
\alpha &= \Pr(X \leq 8 \text{ or } X \geq 21 \mid p = \tfrac{1}{2}) \\
&= \Pr(X \leq 8 \mid p = \tfrac{1}{2}) + [1 - \Pr(X \leq 20 \mid p = \tfrac{1}{2})] \\
&= \texttt{pbinom(8,29,0.5) + 1-pbinom(20,29,0.5)} \\
&= 0.024
\end{aligned}$$

If we used instead *"Reject H_0 if $X \leq 9$ or $X \geq 20$"*, the significance level would be

`pbinom(9,29,0.5) + 1-pbinom(19,29,0.5)` = 0.061

Example 1

Observe X = 24 (for n = 29).

Reject $H_0 : p = \frac{1}{2}$ if $X \leq 8$ or $X \geq 21$.

Thus we reject H_0 and conclude that the ticks were more likely to go to the deer-gland-substance-treated tube.

P-value = $2 \times \Pr(X \geq 24 \mid p = \frac{1}{2})$

= `2*(1-pbinom(23,29,0.5))`

= 0.0005.

$\longrightarrow$ Alternatively: `binom.test(24,29)`

Example 2

Observe X = 17 (for n = 25); assume $X \sim$ Binomial(n=25, p).

Test of $H_0 : p = \frac{1}{2}$ vs $H_a : p \neq \frac{1}{2}$ at significance level α = 0.05.

Rejection rule: Reject H_0 if $X \leq 7$ or $X \geq 18$.

```
qbinom(0.025, 25, 0.5) = 8
pbinom(8, 25, 0.5) = 0.054
pbinom(7, 25, 0.5) = 0.022
```

Significance level:

`pbinom(7,25,0.5) + 1-pbinom(17,25,0.5)` = 0.043

Since we observed X = 17, we fail to reject H_0.

P-value = `2*(1-pbinom(16,25,0.5))` = 0.11

Confidence interval for a proportion

Suppose X $\sim$ Binomial(n=29, p) and we observe X = 24.

Consider the test of $H_0 : p = p_0$ vs $H_a : p \neq p_0$.

We reject H_0 if

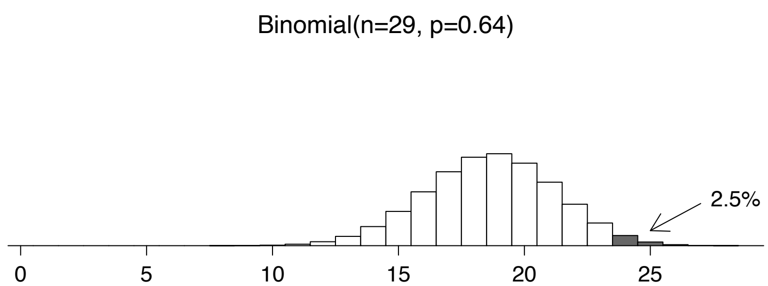
$$\Pr(X \leq 24 \mid p = p_0) \leq \alpha/2 \quad \text{or} \quad \Pr(X \geq 24 \mid p = p_0) \leq \alpha/2$$

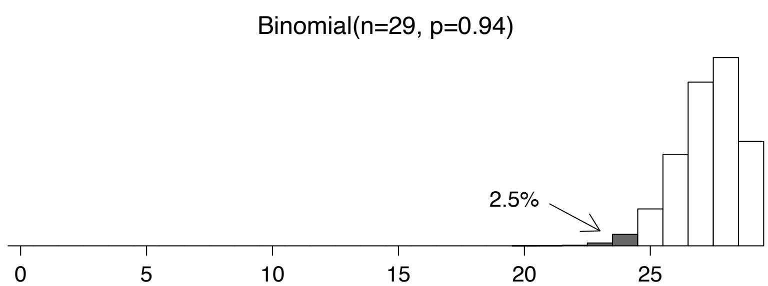
95% confidence interval for p:

$\longrightarrow$ The set of p_0 for which a two-tailed test of $H_0 : p = p_0$ would not be rejected, for the observed data, with α = 0.05.

$\longrightarrow$ The "plausible" values of p.

Example 1





Example 1

$X \sim$ Binomial(n=29, p); observe X = 24.

Lower bound of 95% confidence interval:

Largest p_0 such that $\Pr(X \geq 24 \mid p = p_0) \leq 0.025$

Upper bound of 95% confidence interval:

Smallest p_0 such that $\Pr(X \leq 24 \mid p = p_0) \leq 0.025$

$\longrightarrow$ `binom.test(24,29)`

95% CI for p: (0.642, 0.942)

Note: $\hat{p}$ = 24/29 = 0.83 is not the midpoint of the CI.

Example 2

$X \sim$ Binomial(n=25, p); observe X = 17.

Lower bound of 95% confidence interval:

p_L such that 17 is the 97.5 percentile of Binomial(n=25, p_L)

Upper bound of 95% confidence interval:

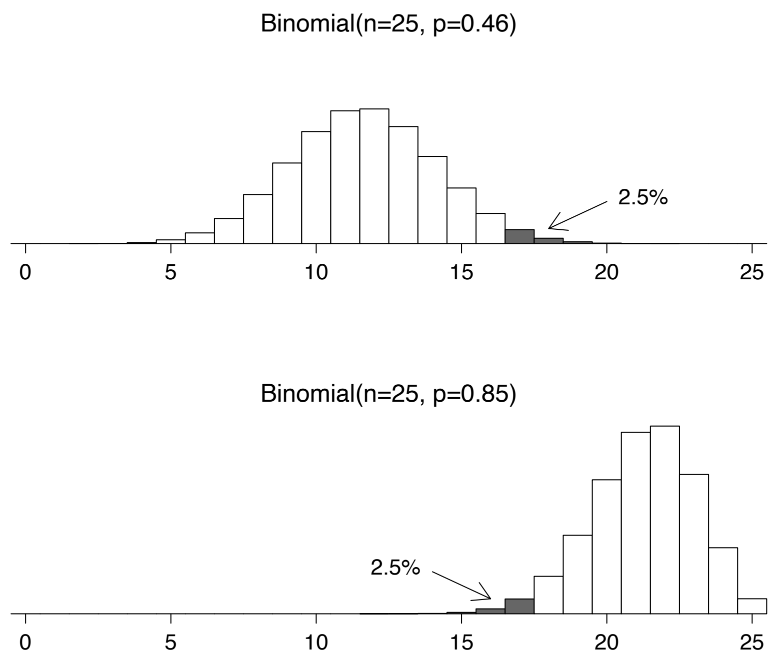
p_H such that 17 is the 2.5 percentile of Binomial(n=25, p_H)

$\longrightarrow$ `binom.test(17,25)`

95% CI for p: (0.465, 0.851)

Again, $\hat{p}$ = 17/25 = 0.68 is not the midpoint of the CI

Example 2



The case X = 0

Suppose X $\sim$ Binomial(n, p) and we observe X = 0.

Lower limit of 95% confidence interval for p: $\to$ 0

Upper limit of 95% confidence interval for p:

p_H such that

$$
\begin{aligned}
& \Pr(X \leq 0 \mid p = p_H) = 0.025 \\
\Longrightarrow\; & \Pr(X = 0 \mid p = p_H) = 0.025 \\
\Longrightarrow\; & (1 - p_H)^n = 0.025 \\
\Longrightarrow\; & 1 - p_H = \sqrt[n]{0.025} \\
\Longrightarrow\; & p_H = 1 - \sqrt[n]{0.025}
\end{aligned}
$$

In the case n = 10 and X = 0, the 95% CI for p is (0, 0.31).

Rule of thumb: a good approximation for the upper bound of the 95% confidence interval is 3/n.

A mad cow example

New York Times, Feb 3, 2004:

> The department [of Agriculture] has not changed last year's plans to test 40,000 cows nationwide this year, out of 30 million slaughtered. Janet Riley, a spokeswoman for the American Meat Institute, which represents slaughterhouses, called that "plenty sufficient from a statistical standpoint."

Suppose that the 40,000 cows tested are chosen at random from the population of 30 million cows, and suppose that 0 (or 1, or 2) are found to be infected.

⟶ How many of the 30 million total cows would we estimate to be infected?

⟶ What is the 95% confidence interval for the total number of infected cows?

No. infected		
Obs'd	Est'd	95% CI
0	0	0 – 2767
1	750	19 – 4178
2	1500	182 – 5418

The case X = n

Suppose $X \sim \text{Binomial}(n, p)$ and we observe $X = n$.

Upper limit of 95% confidence interval for p: $\to 1$

Lower limit of 95% confidence interval for p:

p_L such that

$$\begin{aligned} & \Pr(X \geq n \mid p = p_L) = 0.025 \\ \Longrightarrow\ & \Pr(X = n \mid p = p_L) = 0.025 \\ \Longrightarrow\ & (p_L)^n = 0.025 \\ \Longrightarrow\ & p_L = \sqrt[n]{0.025} \end{aligned}$$

In the case n = 30 and X = 30, the 95% CI for p is (0.88, 1.00).

Rule of thumb: a good approximation for the lower bound of the 95% confidence interval is 1 – 3/n.

Large n and medium p

Suppose X $\sim$ Binomial(n, p).

$$E(X) = n\,p \qquad SD(X) = \sqrt{n\,p(1-p)}$$

$$\hat{p} = X/n \qquad E(\hat{p}) = p \qquad SD(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$$

For large n and medium p, $\longrightarrow$ $\hat{p} \sim \text{Normal}\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$

Use 95% confidence interval $\hat{p} \pm 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

Rule of thumb: if both $n \times \hat{p} > 5$ and $n \times \hat{p} \times (1-\hat{p}) > 5$ this approximation should be OK.

$\longrightarrow$ Unfortunately, this can behave poorly.

$\longrightarrow$ Fortunately, you can just use `binom.test()`